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The Wss P_0 Matrix Completion Problem for Symmetric Patterns of Acyclic Digraphs of Order Four

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Abstract

In this paper we study weakly sign symmetric Wss P_0-matrix specifying symmetric patterns of acyclic digraphs of 4 vertices where necessary and sufficient conditions for a digraphs to have weakly sign symmetric $\mathbf{P_0}$-matrix completion are stated and discussed. It is shown that all symmetric patterns specifying acyclic digraphs of order 4 with without an arc, 2 arcs and regular digraphs with 4 arcs have zero completion to weakly sign symmetric $\mathbf{P_0}$-matrix.

Keywords: Acyclic digraphs; matrix completion; Partial matrix; Principal minor; symmetric patterns; weakly sign symmetric P_0-matrix; Zero completion.

1. Introduction

In this section we define the basic concepts in linear algebra, group theory and graph theory that are commonly used and are fundamental in matrix completion problem.

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Definition 1.1 A **square matrix** A is a matrix that has n rows and n columns i.e. a $n \times n$ matrix. **Principal sub-matrix** $A(\alpha)$ is obtained from $n \times n$ matrix A by deleting all rows and columns not in α, where $\alpha \in \{1,2,...,n\}$. A **principal minor** of A is the determinant of a principal sub-matrix of A [1]. A P_0-**matrix** A is a matrix in which every principal minor of the matrix A is nonnegative. A P_0-matrix A is called a **weakly sign symmetric** P_0 **-matrix** if $a_{ij}a_{ji} \geq 0$ for all i and j. A **partial matrix** is a matrix in which some entries are specified while others are free to be chosen (from a certain set). A partial matrix is a **partial weakly sign symmetric** P_0**-matrix** if determinants of all fully specified principal sub-matrices are nonnegative and $a_{ij}a_{ji} \geq 0$ for all specified entries [1, 2].

Definition 1.2 A **graph** $G = (V_G, E_G)$ is a finite non-empty set of positive integers V_G, whose members are called vertices and a set of E_G (unordered) pairs $\{u,v\}$ of vertices called edges of G. Given a graph $G = (V_G, E_G)$ then a graph $H = (V_H, E_H)$ is a **sub-graph** of graph G if V_H is a subset of V_G and E_H is a subset of E_G. A graph whose edge-set is empty is a **null graph** [3]. A digraph $D = (V_D, E_D)$ is a graph G with ordered pairs (u,v) of vertices and directed edge/arc where u the initial vertex is and v is the terminal vertex. A digraph $H = (V_H, E_H)$) is a **sub-digraph** of digraph D if $V_H \subseteq V_D$ and $E_H \subseteq E_D$. The **Order** of a digraph D denoted $|D|$ is the number of vertices of D. A digraph is **complete** if it includes all possible arcs between its vertices [3, 4].

Definition 1.3 A **Path** P in a digraph D is a sub-digraph of D whose distinct vertices and arcs can be written in an alternating sequence. A closed path is called a **cycle**. A digraph that contains at least one directed cycle is known as a **cycle digraph** while an **acyclic digraph** if it contains no directed cycles. A **degree** of a vertex is the number of edges with that vertex as an end-point. A graph is said to be a **regular graph** if all its vertices have same degree [3].

Definition 1.4 A digraph $D = (V_D, E_D)$ is **isomorphic** to the digraph $D^1 = (V_{D^1}, E_{D^1})$ if there is bijective homomorphism $\phi : V_D \to V_{D^1}$ which is one-to-one and $(u,v) \in E_D$ if and only if $(\phi(u), \theta(v)) \in E_{D^1}$. Two digraphs are said to be isomorphic if their underlying graphs are isomorphic and the direction of the corresponding arcs are same [4].

Definition 1.5 A **pattern** Q for $n \times n$ partial matrices is a list of positions of $n \times n$ matrix that is subset of $\{1,2,...,n\}$ that includes all diagonal positions. A **symmetric pattern** is a pattern with the property that (i, j) is in the pattern if and only if (j,i) is in the pattern while **asymmetric pattern** is a pattern with the property that (i, j) is in the pattern then (j,i) is not in the pattern. A partial matrix **specifies a pattern** if its specified entries lie exactly in those positions listed in the pattern [4].

Definition 1.6 A **completion** of a partial matrix is a specific choice of values for the unspecified entries so that the completed matrix has desired type [5]. Completion of a partial matrix is called **zero completion** if all the unspecified entries in the partial matrix are equated to zeros. A pattern has **weakly sign symmetric P_0-matrix completion** if every partial weakly sign symmetric P_0-matrix that specifies the pattern can be completed to a weakly sign symmetric P_0-matrix [2].

Definition 1.6 A pattern Q is **permutation similar** to a pattern R if there is a permutation ϕ of $\{1,2,\ldots,n\}$ such that $R=\{(\phi(i),\phi(j)):(i,j)\in Q\}$. Relabeling the vertices of a digraph diagram, which performs a digraph isomorphism, corresponds to performing a permutation similarity on the pattern [6].

Lemma 1.7[1] **weakly sign symmetric P_0-matrices are closed under permutation similarity**

Since the class of weakly sign symmetric P_0-matrices is closed under permutation similarity, we are free to relabel digraphs as desired, this implies that the two isomorphic digraphs will have same completion since the determinant is not affected by a permutation similarity.

2. Acyclic digraphs of order 4

In this section, different symmetric patterns for acyclic non-isomorphic digraphs of order 4 are analyzed. Throughout this section we will consider d_i as diagonal entries, a_{ij} as specified entry and x_{ij} as unspecified entry as entries of a partial matrix. We work out the principal minors and apply zero completion i.e. assign all the unspecified entries x_{ij} s to zero and determine if it can be completed to a weakly sign symmetric P_0-matrix.

2.1 Digraph D of order 4 without an arc

Case 2.1.1: Consider a symmetric pattern $Q=\{(1,1),(2,2),(3,3),(4,4)\}$ of a digraph of 4 vertices without an arc given by

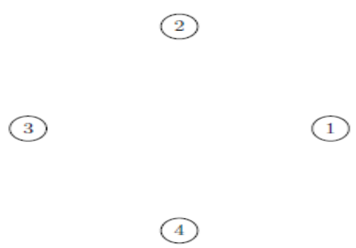

Figure 1: Digraph D of order 4 without an arc

The matrix that specifies the above digraph D is

$$A=\begin{bmatrix} d_1 & x_{12} & x_{13} & x_{14} \\ x_{21} & d_2 & x_{23} & x_{24} \\ x_{31} & x_{32} & d_3 & x_{34} \\ x_{41} & x_{42} & x_{43} & d_4 \end{bmatrix}$$

By definition of partial $Wss\ P_0$-matrix $d_1 \geq 0, d_2 \geq 0, d_3 \geq 0, d_4 \geq 0$

Determinants of Principal Sub-matrices

$$\begin{aligned}
&\det(1) = d_1 \\
&\det(2) = d_2 \\
&\det(3) = d_3 \\
&\det(4) = d_4 \\
&\det(1,2) = d_1 d_2 - x_{12} x_{21} \\
&\det(1,3) = d_1 d_3 - x_{13} x_{31} \\
&\det(1,4) = d_1 d_4 - x_{14} x_{41} \\
&\det(2,3) = d_2 d_3 - x_{23} x_{32} \\
&\det(2,4) = d_2 d_4 - x_{24} x_{42} \\
&\det(3,4) = d_3 d_4 - x_{34} x_{43} \\
&\det(1,2,3) = d_1(d_2 d_3 - x_{23}x_{32}) - x_{12}(d_3 x_{21} - x_{23}x_{31}) + x_{13}(x_{21}x_{32} - d_2 x_{31}) \\
&\det(1,2,4) = d_1(d_2 d_4 - x_{24}x_{42}) - x_{12}(d_4 x_{21} - x_{24}x_{41}) + x_{14}(x_{21}x_{42} - d_2 x_{41}) \\
&\det(1,3,4) = d_1(d_3 d_4 - x_{34}x_{43}) - x_{13}(d_4 x_{31} - x_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_3 x_{41}) \\
&\det(2,3,4) = d_2(d_3 d_4 - x_{34}x_{43}) - x_{23}(d_4 x_{32} - x_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_3 x_{42}) \\
&\det A = d_1[d_2(d_3 d_4 - x_{34}x_{43}) - x_{23}(d_4 x_{32} - x_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_3 x_{42})] \\
&\qquad - x_{12}[x_{21}(d_3 d_4 - x_{34}x_{43}) - x_{23}(d_4 x_{31} - x_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_3 x_{41})] \\
&\qquad + x_{13}[x_{21}(d_4 x_{32} - x_{24}x_{42}) - d_2(d_4 x_{31} - x_{34}x_{41}) + x_{24}(x_{31}x_{42} - x_{32}x_{41})] \\
&\qquad - x_{14}[x_{21}(x_{32}x_{43} - d_3 x_{42}) - d_2(x_{31}x_{43} - d_3 x_{41}) + x_{23}(x_{31}x_{42} - x_{32}x_{41})]
\end{aligned}$$

Zero Completion

Setting all unspecified entries of A to zero

$x_{12} = x_{13} = x_{14} = x_{21} = x_{23} = x_{24} = x_{31} = x_{32} = x_{34} = x_{41} = x_{42} = x_{43} = 0$, gives

$$\det(1) = d_1 \geq 0, \quad \det(2) = d_2 \geq 0, \quad \det(3) = d_3 \geq 0, \quad \det(4) = d_4 \geq 0$$

$$\det(1,2) = d_1 d_2 \geq 0, \quad \det(1,3) = d_1 d_3 \geq 0, \quad \det(1,4) = d_1 d_4 \geq 0$$
$$\det(2,3) = d_2 d_3 \geq 0, \quad \det(2,4) = d_2 d_4 \geq 0, \quad \det(3,4) = d_3 d_4 \geq 0$$

$$\det(1,2,3) = d_1 d_2 d_3 \geq 0, \quad \det(1,2,4) = d_1 d_2 d_4 \geq 0,$$
$$\det(1,3,4) = d_1 d_3 d_4 \geq 0, \quad \det(2,3,4) = d_2 d_3 d_4 \geq 0,$$
$$\det A = d_1 d_2 d_3 d_4 \geq 0$$

Since all the determinants nonnegative then the partial matrix is completed to $Wss\ P_0$-matrix. Hence it has a zero completion into a $Wss\ P_0$-matrix.

2.2 Digraph D of order 4 and 2 arcs

Case 2.2.1: Consider a symmetric pattern $Q = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$ of a digraph of 4 vertices and 2 arcs given by

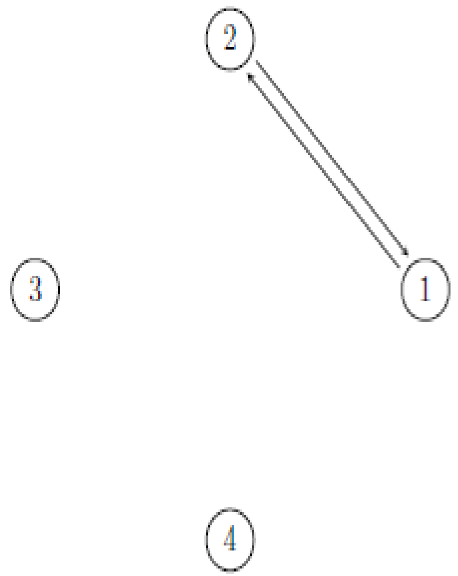

Figure 2: Digraph D of order 4 with 2 arcs

The matrix that specifies the above digraph D is $A = \begin{bmatrix} d_1 & a_{12} & x_{13} & x_{14} \\ a_{21} & d_2 & x_{23} & x_{24} \\ x_{31} & x_{32} & d_3 & x_{34} \\ x_{41} & x_{42} & x_{43} & d_4 \end{bmatrix}$

By definition of partial $Wss\ P_0$-matrix $d_1 \geq 0, d_2 \geq 0, d_3 \geq 0, d_4 \geq 0$

Determinants of Principal Sub-matrices

$$
\begin{aligned}
&\det(1) = d_1 \\
&\det(2) = d_2 \\
&\det(3) = d_3 \\
&\det(4) = d_4 \\
&\det(1,2) = d_1 d_2 - a_{12} a_{21} \\
&\det(1,3) = d_1 d_3 - x_{13} x_{31} \\
&\det(1,4) = d_1 d_4 - x_{14} x_{41} \\
&\det(2,3) = d_2 d_3 - x_{23} x_{32} \\
&\det(2,4) = d_2 d_4 - x_{24} x_{42} \\
&\det(3,4) = d_3 d_4 - x_{34} x_{43} \\
&\det(1,2,3) = d_1 (d_2 d_3 - x_{23} x_{32}) - a_{12}(d_3 a_{21} - x_{23} x_{31}) + x_{13}(a_{21} x_{32} - d_2 x_{31}) \\
&\det(1,2,4) = d_1 (d_2 d_4 - x_{24} x_{42}) - a_{12}(d_4 a_{21} - x_{24} x_{41}) + x_{14}(a_{21} x_{42} - d_2 x_{41}) \\
&\det(1,3,4) = d_1 (d_3 d_4 - x_{34} x_{43}) - x_{13}(d_4 x_{31} - x_{34} x_{41}) + x_{14}(x_{31} x_{43} - d_3 x_{41}) \\
&\det(2,3,4) = d_2 (d_3 d_4 - x_{34} x_{43}) - x_{23}(d_4 x_{32} - x_{34} x_{42}) + x_{24}(x_{32} x_{43} - d_3 x_{42}) \\
&\det A = d_1 [d_2 (d_3 d_4 - x_{34} x_{43}) - x_{23}(d_4 x_{32} - x_{34} x_{42}) + x_{24}(x_{32} x_{43} - d_3 x_{42})] \\
&\qquad - a_{12}[a_{21}(d_3 d_4 - x_{34} x_{43}) - x_{23}(d_4 x_{31} - x_{34} x_{41}) + x_{24}(x_{31} x_{43} - d_3 x_{41})] \\
&\qquad + x_{13}[a_{21}(d_4 x_{32} - x_{24} x_{42}) - d_2 (d_4 x_{31} - x_{34} x_{41}) + x_{24}(x_{31} x_{42} - x_{32} x_{41})] \\
&\qquad - x_{14}[a_{21}(x_{32} x_{43} - d_3 x_{42}) - d_2 (x_{31} x_{43} - d_3 x_{41}) + x_{23}(x_{31} x_{42} - x_{32} x_{41})]
\end{aligned}
$$

Zero Completion

Setting all unspecified entries of A to zero

$x_{13} = x_{14} = x_{23} = x_{24} = x_{31} = x_{32} = x_{34} = x_{41} = x_{42} = x_{43} = 0$, gives

$$\det(1) = d_1 \geq 0, \quad \det(2) = d_2 \geq 0, \quad \det(3) = d_3 \geq 0, \quad \det(4) = d_4 \geq 0$$

$\det(1,2) = d_1 d_2 - a_{12} a_{21} \geq 0,$ (Since $(1,2)$ is fully specified)

$$\det(1,3) = d_1 d_3 \geq 0, \quad \det(1,4) = d_1 d_4 \geq 0$$
$$\det(2,3) = d_2 d_3 \geq 0, \quad \det(2,4) = d_2 d_4 \geq 0, \quad \det(3,4) = d_3 d_4 \geq 0$$

$\det(1,2,3) = d_1 d_2 d_3 - d_3 a_{12} a_{21} = d_3 (d_1 d_2 - a_{12} a_{21}) \geq 0,$ (Since $(1,2)$ is fully specified)

$\det(1,2,4) = d_1 d_2 d_4 - d_4 a_{12} a_{21} = d_4 (d_1 d_2 - a_{12} a_{21}) \geq 0,$ (Since $(1,2)$ is fully specified)

$$\det(1,3,4) = d_1 d_3 d_4 \geq 0, \quad \det(2,3,4) = d_2 d_3 d_4 \geq 0,$$

$\det A = d_1 d_2 d_3 d_4 - d_3 d_4 a_{12} a_{21} = d_3 d_4 (d_1 d_2 - a_{12} a_{21}) \geq 0,$ (Since $(1,2)$ is fully specified)

Since all the determinants nonnegative then the partial matrix is completed to $Wss\ P_0$-matrix. Hence it has a zero completion into a $Wss\ P_0$-matrix.

2.3 Digraph D of order 4 and 4 arcs

Case 2.3.1: Consider a symmetric pattern $Q = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3),(4,4)\}$ of a digraph of 4 vertices and 4 arcs given by

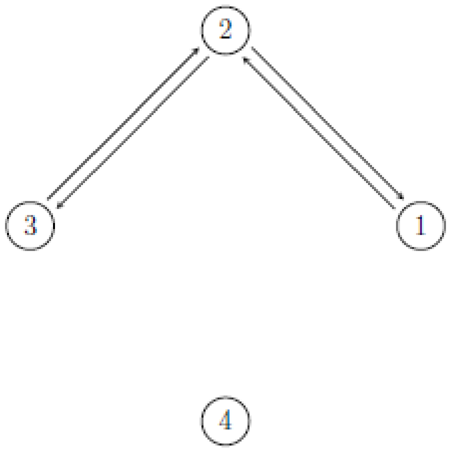

Figure 3: Digraph D of order 4 with 4 arcs

The matrix that specifies the above digraph D is $A = \begin{bmatrix} d_1 & a_{12} & x_{13} & x_{14} \\ a_{21} & d_2 & a_{23} & x_{24} \\ x_{31} & a_{32} & d_3 & x_{34} \\ x_{41} & x_{42} & x_{43} & d_4 \end{bmatrix}$

By definition of partial $Wss\ P_0$-matrix $d_1 \geq 0, d_2 \geq 0, d_3 \geq 0, d_4 \geq 0$

Similar to case 2.1.1 and 2.2.1 finding the determinants of Principal Sub-matrices and performing zero completion; Setting all unspecified entries of A to zero, $x_{13} = x_{14} = x_{24} = x_{31} = x_{34} = x_{41} = x_{42} = x_{43} = 0$, gives

$\det(1) = d_1 \geq 0, \quad \det(2) = d_2 \geq 0, \quad \det(3) = d_3 \geq 0, \quad \det(4) = d_4 \geq 0$

$\det(1,2) = d_1 d_2 - a_{12} a_{21} \geq 0,$ (Since $(1,2)$ is fully specified)

$\det(1,3) = d_1 d_3 \geq 0, \quad \det(1,4) = d_1 d_4 \geq 0$

$\det(2,3) = d_2 d_3 - a_{23} a_{32} \geq 0,$ (Since $(2,3)$ is fully specified)

$\det(2,4) = d_2 d_4 \geq 0, \quad \det(3,4) = d_3 d_4 \geq 0$

$\det(1,2,3) = d_1 d_2 d_3 - d_1 a_{23} a_{32} - d_3 a_{12} a_{21} = d_3 (d_1 d_2 - a_{12} a_{21}) - d_1 a_{23} a_{32},$
$\det(1,2,4) = d_1 d_2 d_4 - d_4 a_{12} a_{21} = d_4 (d_1 d_2 - a_{12} a_{21}) \geq 0,$ (Since $(1,2)$ is fully specified)

$\det(1,3,4) = d_1 d_3 d_4 \geq 0,$

$\det(2,3,4) = d_2 d_3 d_4 - d_4 a_{23} a_{32} = d_4 (d_2 d_3 - a_{23} a_{32}) \geq 0,$ (Since $(2,3)$ is fully specified)

$\det A = d_1 d_2 d_3 d_4 - d_1 d_4 a_{23} a_{32} - d_3 d_4 a_{12} a_{21} = d_3 d_4 (d_1 d_2 - a_{12} a_{21}) - d_1 d_4 a_{23} a_{32}$

Since all the determinants are not nonnegative then the partial matrix cannot be completed to a $Wss\ P_0$-matrix. Hence it has no zero completion into a $Wss\ P_0$-matrix.

Case 2.3.2: Consider a symmetric pattern $Q = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4)\}$ of a digraph of 4 vertices and 4 arcs given by

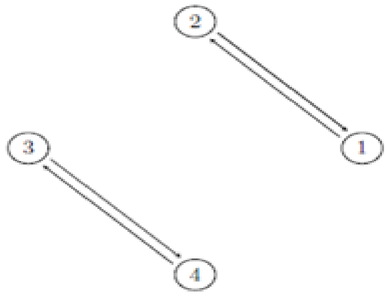

Figure 4: Digraph D of order 4 with 4 arcs

The matrix that specifies the above digraph D is $A = \begin{bmatrix} d_1 & a_{12} & x_{13} & x_{14} \\ a_{21} & d_2 & x_{23} & x_{24} \\ x_{31} & x_{32} & d_3 & a_{34} \\ x_{41} & x_{42} & a_{43} & d_4 \end{bmatrix}$

By definition of partial $Wss\ P_0$-matrix $d_1 \geq 0, d_2 \geq 0, d_3 \geq 0, d_4 \geq 0$

Finding the determinants of Principal Sub-matrices and performing zero completion; Setting all unspecified entries of A to zero, $x_{13} = x_{14} = x_{23} = x_{24} = x_{31} = x_{32} = x_{41} = x_{42} = 0$, gives

$\det(1) = d_1 \geq 0, \quad \det(2) = d_2 \geq 0, \quad \det(3) = d_3 \geq 0, \quad \det(4) = d_4 \geq 0$

$\det(1,2) = d_1 d_2 - a_{12} a_{21} \geq 0,$ (Since $(1,2)$ is fully specified)

$\det(1,3) = d_1 d_3 \geq 0, \quad \det(1,4) = d_1 d_4 \geq 0$

$\det(2,3) = d_2 d_3 \geq 0, \quad \det(2,4) = d_2 d_4 \geq 0$

$\det(3,4) = d_3 d_4 - a_{34} a_{43} \geq 0,$ (Since $(3,4)$ is fully specified)

$\det(1,2,3) = d_1 d_2 d_3 - d_3 a_{12} a_{21} = d_3 (d_1 d_2 - a_{12} a_{21}) \geq 0,$ (Since $(1,2)$ is fully specified)

$\det(1,2,4) = d_1 d_2 d_4 - d_4 a_{12} a_{21} = d_4 (d_1 d_2 - a_{12} a_{21}) \geq 0,$ (Since $(1,2)$ is fully specified)

$\det(1,3,4) = d_1 d_3 d_4 - d_1 a_{34} a_{43} = d_1 (d_3 d_4 - a_{34} a_{43}) \geq 0,$ (Since $(3,4)$ is fully specified)

$\det(2,3,4) = d_2 d_3 d_4 - d_2 a_{34} a_{43} = d_2 (d_3 d_4 - a_{34} a_{43}) \geq 0,$ (Since $(3,4)$ is fully specified)

$\det A = d_1 d_2 d_3 d_4 - d_1 d_2 a_{34} a_{43} - d_3 d_4 a_{12} a_{21} + a_{12} a_{21} a_{34} a_{43} = (d_1 d_2 - a_{12} a_{21})(d_3 d_4 - a_{34} a_{43}) \geq 0,$
(Since $(1,2)$ and $(3,4)$ are fully specified)

Since all the determinants are nonnegative then the partial matrix can be completed to a $Wss\ P_0$-matrix. Hence it has a zero completion into a $Wss\ P_0$-matrix.

2.4 Digraph D of order 4 and 6 arcs

The symmetric patterns of acyclic digraphs of order 4 and 6 arcs are:

Case 2.4.1 $Q_1 = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3),(3,4),(4,3),(4,4)\}$

Case 2.4.2 $Q_2 = \{(1,1),(1,2),(2,1),(2,2),(2,3),(2,4),(3,2),(3,3),(4,2),(4,4)\}$

The matrices that specifies Q_1 and Q_2 are:

$$A_1 = \begin{bmatrix} d_1 & a_{12} & x_{13} & x_{14} \\ a_{21} & d_2 & a_{23} & x_{24} \\ x_{31} & a_{32} & d_3 & a_{34} \\ x_{41} & x_{42} & a_{43} & d_4 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} d_1 & a_{12} & x_{13} & x_{14} \\ a_{21} & d_2 & a_{23} & a_{24} \\ x_{31} & a_{32} & d_3 & x_{34} \\ x_{41} & a_{42} & x_{43} & d_4 \end{bmatrix} \text{ respectively.}$$

Finding the determinants of the principal sub-matrices and performing zero completion shows that all the determinants are not nonnegative then the partial matrix cannot be completed to a $Wss\ P_0$-matrix. Hence it both cases of the patterns have no zero completion into a $Wss\ P_0$-matrix.

3. Conclusion and Recommendation

In this paper, it was concluded that all symmetric patterns specifying acyclic digraphs of order 4 without an arc (null graph), 2 arcs and regular acyclic digraphs with 4 arcs have a zero completion into a weakly sign symmetric $Wss\ P_0$-matrix. Similar research should be done for symmetric patterns specifying cyclic digraphs and even those patterns that are neither symmetric nor asymmetric patterns which are specifying acyclic or cyclic digraphs of order 4

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