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Property of Fourier Transforms with ω-Shifting and x-Shifting

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Abstract

The authors establish a set of presumably new properties. If we have the Fourier transform of $f(x)$, this property helps us to get immediately the Fourier transform of $e^{ax}f(x)$. Also complicated inputs $r(t)$ (right sides of linear differential equations) can be handled very efficiently and Heaviside shall drop variables when this simplifies formulas without causing confusion by using this properties [3].

Key words and phrases: Fourier transforms Heaviside functions.

1. Introduction

On 21 December 1807, in one of the most memorable sessions of the French Academy, Jean Baptiste Joseph Fourier, a 21-year old Mathematician and engineer announced a thesis which began a new chapter in the history of Mathematics. Fourier claimed that an arbitrary function, defined in a finite interval by an arbitrary and capricious graph, can always be resolved into a sum of pure sine and cosine He wanted to use this form to come up with solution to certain linear partial differential equations (specifically the heat equation) because sines and cosines behave nicely under differentiation. For instance, he said the derivative of the above functions should be this sum. 1828 Dirichlet formulated conditions for a function f(x) to have the Fourier transform $f(x)$ must be single valued have a finite number of discontinuities in any given interval have a finite number of extrema in any given interval be square-integrable.

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Fourier transforms can be used as tools is solving ODEs, PDEs and integral equations can be often help in handling and applying special functions [1,2,9,10].

1.1. Definition

The Fourier transform of the function $f(x)$ is given by:

$$\hat{f}(\omega) = \mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

[4,5,6,7,8].

2. Property Fourier Transforms With ω-Shifting and x-Shifting

The Fourier transform has the very useful property that, if we know the transform of $f(x)$ we can immediately get that of $e^{ax} f(x)$, as follows.

Theorem 1: (Property of Fourier Transforms with ω-Shifting)

If $f(x)$ has Fourier transform b $\hat{f}(\omega)$ (where! $\omega > k$,) for some k), then has the Fourier transform b $\hat{f}(\omega)$ (where $\omega - a > k$). In formulas,

$$\hat{f}\left(e^{ax} f(x)\right) = \hat{f}(\omega - a)$$

or, if we take the inverse on both sides,

$$e^{ax} f(x) = \hat{f}^{-1}(\hat{f}(\omega - a))$$

Proof

We obtain $\hat{f}(\omega - a)$ by replacing with ω by $\omega - a$ in the definition Fourier transforms,

so that

$$\hat{f}(\omega - a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(\omega - a)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} e^{iax} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} [f(x) e^{-i\omega x}] dx$$

$$= \hat{f}\left(e^{ax} f(x)\right)$$

Fourier Transforms and Unit Step Function (Heaviside Function)

The unit step function or Heaviside function $u(x - a)$ is:

$$\mathcal{F}\left(u(x-a)\right) = \begin{cases} 0, & if\ \ x < a \\ 1, & if\ \ x > a \end{cases}, \qquad a \geq 0$$

Then,

$$\mathcal{F}\left(u(x-a)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x-a) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-i\omega x} dx$$

$$= \lim_{k \to \infty} [\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{k} e^{-i\omega x} dx]$$

$$= \lim_{k \to \infty} [\frac{1}{\sqrt{2\pi}} \frac{-1}{i\omega} e^{-ik\omega} + \frac{1}{\sqrt{2\pi}} \frac{1}{i\omega} e^{-ia\omega}]$$

$$= \frac{1}{i\omega\sqrt{2\pi}} e^{-ia\omega}$$

Hence,

$$\mathcal{F}\left(u(x-a)\right) = \frac{1}{i\omega\sqrt{2\pi}} e^{-ia\omega}$$

Let $f(x) = 0$ for all negative x. Then $f(x - a)u(x - a)$ with $a > 0$ is shifted (translated) to the right by the amount a.

Theorem 2: (Property of Fourier Transforms with x-Shifting)

If $f(x)$ has the Fourier transform then the shifted function

$$\bar{f}(x) = f(x-a)u(x-a) = \begin{cases} 0, & if\ \ x < a \\ f(x-a), & if\ \ x > a \end{cases}, \qquad a \geq 0$$

has the Fourier transform $e^{ax}\hat{f}(\omega)$ That is, if F $\mathcal{F}\left(f(x)\right) = \hat{f}(\omega)$, then

$$\mathcal{F}\big(f(x-a)u(x-a)\big) = e^{-ia\omega}\mathcal{F}\big(f(x)\big)$$

Or, if we take the inverse on both sides, we can write

$$f(x-a)u(x-a) = \mathcal{F}^{-1}[e^{-ia\omega}\mathcal{F}\big(f(x)\big)]$$

Proof

By using the definition of Fourier transforms and writing t for x. Then taking

$e^{-ia\omega}$ inside the integral, we have

$$e^{-ia\omega}\hat{f}(\omega) = e^{-ia\omega}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega(t+a)}dt$$

Substituting $t + a = x$, thus $t = x - a$; $dt = dx$ in the integral

$$e^{-ia\omega}\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{-i\omega x}dx$$

$$= \mathcal{F}\big(f(x-a)u(x-a)\big)$$

Therefore,

$$\mathcal{F}\big(f(x-a)u(x-a)\big) = e^{-ia\omega}\mathcal{F}\big(f(x)\big)$$

3. Conclusions

The properties is very essential in finding $e^{ax}f(x)$, if we have the Fourier transform of $f(x)$. Also complicated inputs $r(t)$ (right sides of linear differential equations) can be handled very efficiently and Heaviside shall drop variables when this simplifies formulas without causing confusion by using this properties.

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