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Properties of Fourier Cosine and Sine Integrals with the Product of Power and Polynomial Functions

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Abstract

The time and frequency domains are alternative ways of representing signals. The Fourier transform is the mathematical relationship between these two representations. These transformations are of interest mainly as tools for solving ODEs, PDEs and integral equations, and they often also help in handling and applying special functions. In this article, I have outlined the main features of properties of Fourier cosine and sine Integrals. These properties demand the implementation of representation of a function in integral form, known as Fourier cosine and sine integrals. The purpose of this paper is to provide a brief representation any function in integral form, Fourier cosine and sine transforms, after multiplying the given function by power functions and polynomials and provide the relation between Fourier Cosine integrals and Fourier Sine integrals [9,10].

Keywords: Fourier integrals; Fourier cosine and sine integrals.

1. Introduction

On 21 December 1807, in one of the most memorable sessions of the French Academy, Jean Baptiste Joseph Fourier, a 21-year old Mathematician and engineer announced a thesis which began a new chapter in the history of Mathematics. Fourier claimed that an arbitrary function, defined in a finite interval by an arbitrary and capricious graph, can always be resolved into a sum of pure sine and cosine He wanted to use this form to come up with solution to certain linear partial differential equations (specifically the heat equation) because sines and cosines behave nicely under differentiation.

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For instance, he said the derivative of the above functions should be this sum: 1828 Dirichlet formulated conditions for a function f(x) to have the Fourier transform f(x) must be single valued have a finite number of discontinuities in any given interval have a finite number of extrema in any given interval be square-integrable. . [1,2,3,8].

1.1. Definition

The Fourier cosine integral of the function $f\ (x\ )$ is given by

$$f(x) = \int_0^\infty A(\omega)\cos x\omega d\omega$$

Where,

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos x\omega dx$$

[3,4,5,6]:

1.2. Definition

The Fourier sine integral of the function $f(x)$ is given by:

$$f(x) = \int_0^\infty B(\omega)\sin x\omega d\omega$$

Where,

$$B(x) = \frac{2}{\pi}\int_0^\infty f(x)\sin x\omega dx$$

[4,5,6,7]:

1.3. Definition

The Fourier integral of the function $f(x)$ is given by:

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(\omega)\cos x\omega + B(\omega)\sin x\omega] d\omega$$

Where,

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos x\omega\, dx$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin x\omega\, dx$$

[4,5,6,7]:

2. Properties of Fourier Cosine and Sine Integrals with a product of power functions

Theorem 1: (Properties of Cosine and Sine Integrals with a product of power functions)

A) If $f(x)$ has a Fourier cosine integral:

$$f(x)=\int_{0}^{\infty} A(\omega)\cos x\omega d\omega$$

Where,

$$A(\omega)=\frac{2}{\pi}\int_{0}^{\infty} f(x)\cos x\omega dx$$

Then, the Fourier cosine integral of :

$$xf(x) \;= f(x)=\int_{0}^{\infty} A_1(\omega)\cos x\omega d\omega$$

Such that, $A_1(\omega)=\frac{d}{d\omega}(B(\omega))$

$$x^2 f(x) \;=\int_{0}^{\infty} A_2(\omega)\cos x\omega d\omega$$

Such that, $A_2(\omega)=-\frac{d^2}{d\omega^2}(A(\omega))$

In general, for an even natural number n, the Fourier cosine integral of a new function $x^n f(x)$ is:

$$x^n f(x) \;=\int_{0}^{\infty} A_n(\omega)\cos x\omega d\omega$$

Such that,

$$A_n(\omega)=(-1)^{\frac{n}{2}}\frac{d^n}{d\omega^n}[A(\omega)]$$

For an odd natural number n, the Fourier cosine integral of new function $x^n f(x)$ is:

$$x^n f(x) = \int_0^\infty A_n(\omega)\cos x\omega d\omega$$

Such that,

$$A_n(\omega) = (-1)^{\left(\frac{n+3}{2}\right)} \frac{d^n}{d\omega^n}[B(\omega)]$$

Proof

Suppose $f(x)$ has a Fourier cosine and sine integrals:

$$f(x) = \int_0^\infty A(\omega)\cos x\omega d\omega$$

Where,

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos x\omega dx$$

And

$$f(x) = \int_0^\infty B(\omega)\sin x\omega d\omega$$

Where,

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin x\omega dx$$

Let, multiply $f(x)$ by x, then by definition; the Fourier cosine integral of $xf(x)$ is:

$$xf(x) = \int_0^\infty A_1(\omega)\cos x\omega d\omega$$

Where,

$$A_1(\omega) = \frac{2}{\pi}\int_0^\infty xf(x)\cos x\omega dx$$

Then, from the Fourier sine integral we have:

$$f(x) = \int_0^\infty B(\omega) \sin x\omega d\omega$$

Where,

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(x) \sin x\omega dx$$

By taking differentiation to both sides with respect to $'\omega'$;

$$\frac{d}{d\omega}(B(\omega)) = \frac{\mathrm{d}}{\mathrm{d}\omega}(\frac{2}{\pi}\int_0^\infty f(x) \sin x\omega dx)$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\omega}(f(x) \sin x\omega) dx$$

$$= \frac{2}{\pi}\int_0^\infty f(x) \frac{\mathrm{d}}{\mathrm{d}\omega}(\sin x\omega) dx$$

$$= \frac{2}{\pi}\int_0^\infty x f(x) \cos x\omega dx$$

$$= A_1(\omega)$$

Therefore,

$$A_1(\omega) = \frac{d}{d\omega}(B(\omega))$$

Let, multiply $f(x)$ by x^2, then by definition: The Fourier cosine integral of $x^2 f(x)$ is:

$$x^2 f(x) = \int_0^\infty A_2(\omega) \cos x\omega d\omega$$

Where,

$$A_2(\omega) = \frac{2}{\pi}\int_0^\infty x^2 f(x) \cos x\omega dx$$

From the definition of Fourier cosine integral we have:

$$f(x) = \int_0^\infty A(\omega)\cos x\omega d\omega$$

Where,

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos x\omega dx$$

By taking differentiation to both sides with respect $'\omega'$ twice, we have:

$$\frac{d}{d\omega}(A(\omega)) = \frac{d}{d\omega}(\frac{2}{\pi}\int_0^\infty f(x)\cos x\omega dx)$$

$$= \frac{2}{\pi}\int_0^\infty f(x)\frac{d}{d\omega}(\cos x\omega)dx$$

$$= \frac{2}{\pi}\int_0^\infty -xf(x)\sin x\omega dx$$

$$= -\frac{2}{\pi}\int_0^\infty xf(x)\sin x\omega dx$$

And

$$\frac{d^2}{d\omega^2}(A(\omega)) = \frac{d^2}{d\omega^2}(\frac{2}{\pi}\int_0^\infty f(x)\cos x\omega dx)$$

$$= \frac{d}{d\omega}[\frac{\mathrm{d}}{\mathrm{d\omega}}\left(\frac{2}{\pi}\int_0^\infty f(x)\cos x\omega dx\right)]$$

$$= \frac{d}{d\omega}(-\frac{2}{\pi}\int_0^\infty xf(x)\sin x\omega dx)$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{d}{d\omega}[xf(x)\sin x\omega dx]$$

$$= -\frac{2}{\pi}\int_0^\infty x^2 f(x)\cos x\omega dx$$

$$= -A_2(\omega)$$

Therefore,

$$A_2(\omega) = \frac{d^2}{d\omega^2}(A(\omega))$$

Continuing the process, for an even natural number n,

$$A_n(\omega) = (-1)^{\left(\frac{n}{2}\right)}\frac{B^n}{d\omega^n}(A(\omega))$$

Such that, the Fourier cosine integral of $x^n f(x)$ is:

$$x^n f(x) = \int_0^\infty A_n(\omega)\cos x\omega d\omega$$

And

For an odd natural number n,

$$A_n(\omega) = (-1)^{\left(\frac{n+3}{2}\right)}\frac{d^n}{d\omega^n}(B(\omega))$$

Such that, the Fourier cosine integral of $x^n f(x)$ is:

$$x^n f(x) = \int_0^\infty A_n(\omega)\cos x\omega d\omega$$

B) If $f(x)$ has a Fourier sine integral:

$$f(x) = \int_0^\infty B(\omega)\sin x\omega d\omega$$

Where,

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin x\omega dx$$

Then, the Fourier sine integral of :

$$xf(x) = \int_0^\infty B_1(\omega) \sin x\omega d\omega$$

Such that, $B_1(\omega) = -\frac{d}{d\omega}[A(\omega)]$

$$x^2 f(x) = \int_0^\infty B_2(\omega) \sin x\omega d\omega$$

Such that, $B_2(\omega) = -\frac{d^2}{d\omega^2}[B(\omega)]$

In general, for an even natural number n, the Fourier sine integral of a new function $x^n f(x)$ is:

$$x^n f(x) = \int_0^\infty B_n(\omega) \sin x\omega d\omega$$

Such that,

$$B_n(\omega) = (-1)^{\left(\frac{n}{2}\right)} \frac{d^n}{d\omega^n}[B(\omega)]$$

For an odd natural number n, the Fourier cosine integral of new function $x^n f(x)$ is:

$$x^n f(x) = \int_0^\infty B_n(\omega) \sin x\omega d\omega$$

Such that,

$$B_n(\omega) = (-1)^{\left(\frac{n+3}{2}\right)} \frac{d^n}{d\omega^n}[A(\omega)]$$

Proof

Suppose $f(x)$ has a Fourier cosine and sine integrals:

$$f(x) = \int_0^\infty A(\omega) \cos x\omega d\omega$$

Where,

$$A(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \cos x\omega dx$$

And

$$f(x) = \int_0^\infty B(\omega) \sin x\omega d\omega$$

Where,

$$B(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin x\omega dx$$

Let, multiply $f(x)$ by x, then by definition; the Fourier cosine integral of $xf(x)$ is:

$$xf(x) = \int_0^\infty B_1(\omega) \sin x\omega d\omega$$

Where,

$$B_1(\omega) = \frac{2}{\pi} \int_0^\infty xf(x) \sin x\omega dx$$

Then, from the Fourier cosine integral we have:

$$f(x) = \int_0^\infty A(\omega) \cos x\omega d\omega$$

Where,

$$A(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \cos x\omega dx$$

By taking differentiation to both sides with respect to $'\omega'$;

$$\frac{d}{d\omega}(A(\omega)) = \frac{\mathrm{d}}{\mathrm{d}\omega}(\frac{2}{\pi} \int_0^\infty f(x) \cos x\omega dx)$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\omega}(f(x) \cos x\omega) dx$$

$$= \frac{2}{\pi} \int_0^\infty f(x) \frac{\mathrm{d}}{\mathrm{d}\omega}(\cos x\omega) dx$$

$$= \frac{2}{\pi}\int_{0}^{\infty} -xf(x)\sin x\omega dx$$

$$= -\frac{2}{\pi}\int_{0}^{\infty} xf(x)\sin x\omega dx$$

$$= -B_1(\omega)$$

Therefore,

$$B_1(\omega) = -\frac{d}{d\omega}(A(\omega))$$

Let, multiply $f(x)$ by x^2, then by definition: The Fourier sine integral of $x^2 f(x)$ is:

$$x^2 f(x) = \int_{0}^{\infty} B_2(\omega)\sin x\omega d\omega$$

Where,

$$B_2(\omega) = \frac{2}{\pi}\int_{0}^{\infty} x^2 f(x)\ \sin x\omega dx$$

From the definition of Fourier sine integral we have:

$$f(x) = \int_{0}^{\infty} B(\omega)\sin x\omega d\omega$$

Where,

$$B(\omega) = \frac{2}{\pi}\int_{0}^{\infty} f(x)\sin x\omega dx$$

By taking differentiation to both sides with respect $'\omega'$ twice, we have:

$$\frac{d}{d\omega}(B(\omega)) = \frac{d}{d\omega}(\frac{2}{\pi}\int_{0}^{\infty} f(x)\sin x\omega dx)$$

$$= \frac{2}{\pi}\int_{0}^{\infty} f(x)\frac{d}{d\omega}(\sin x\omega)dx$$

$$= \frac{2}{\pi}\int_0^\infty xf(x)\cos x\omega dx$$

And

$$\frac{d^2}{d\omega^2}(A(\omega)) = \frac{d^2}{d\omega^2}(\frac{2}{\pi}\int_0^\infty f(x)\sin x\omega dx)$$

$$= \frac{d}{d\omega}[\frac{\mathrm{d}}{\mathrm{d\omega}}\left(\frac{2}{\pi}\int_0^\infty f(x)\sin x\omega dx\right)]$$

$$= \frac{d}{d\omega}(\frac{2}{\pi}\int_0^\infty xf(x)\cos x\omega dx)$$

$$= \frac{2}{\pi}\int_0^\infty \frac{d}{d\omega}[xf(x)\cos x\omega dx]$$

$$= -\frac{2}{\pi}\int_0^\infty x^2 f(x)\sin x\omega dx$$

$$= -B_2(\omega)$$

Therefore,

$$B_2(\omega) = -\frac{d^2}{d\omega^2}(B(\omega))$$

Continuing the process, for an even natural number n,

$$B_n(\omega) = (-1)^{\left(\frac{n}{2}\right)}\frac{d^n}{d\omega^n}(B(\omega))$$

Such that, the Fourier cosine integral of $x^n f(x)$ is:

$$x^n f(x) = \int_0^\infty B_n(\omega)\sin x\omega d\omega$$

And

For an odd natural number n,

$$B_n(\omega) = (-1)^{\left(\frac{n+3}{2}\right)} \frac{d^n}{d\omega^n}(A(\omega))$$

Such that, the Fourier cosine integral of $x^n f(x)$ is:

$$x^n f(x) = \int_0^\infty B_n(\omega) \sin x\omega d\omega$$

3. Properties of Fourier Cosine and Sine Integrals with a product of polynomial functions

Theorem 2: (Properties of Cosine and Sine Integrals with a product of polynomial functions)

A) Consider a polynomial function:

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

with real numbers coefficients and assume that f(x) has Fourier cosine integral. Then, for any natural number n, the Fourier cosine integral of a new function, $P_n(x)f(x)$, given by:

$$P_n(x)f(x) = \int_0^\infty A_n{}^*(\omega) \cos x\omega d\omega$$

Such that,

$$A_n{}^*(\omega) = \sum_{m=0}^{n} a_n(-1)^{\left(\frac{n}{2}\right)} \frac{d^n}{d\omega^n}\big(A(\omega)\big) + \sum_{r=0}^{n-1} a_{n-1}(-1)^{\left(\frac{n+3}{2}\right)} \frac{d^{n-1}}{d\omega^{n-1}}\big(B(\omega)\big)$$

Where, $m = 0, 2, 4, 6, \ldots, n \quad and \;\; r = 1, 3, 5, \ldots, n-1$

Proof

Suppose $f(x)$ has a Fourier cosine integral and $P_n(x)$ be any polynomial function of degree n:

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

$$= a_n x^n + +a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0 + a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_1 x$$

$$= g(x) + h(x)$$

Where, $g(x) = a_n x^n + +a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0$ implies all powers of variables are even and $h(x) = a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_1 x$ with all powers of variables are odd.

Then, the Fourier cosine integral of a new function $g(x)f(x)$ is given by:

$$g(x)f(x) = \int_0^\infty {A_n}^1(\omega) \cos x\omega d\omega$$

Where,

$${A_n}^1(\omega) = \frac{2}{\pi}\int_0^\infty g(x)f(x) \cos x\omega dx$$

$$= \frac{2}{\pi}\int_0^\infty [a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0] f(x) \cos x\omega dx$$

$$= \frac{2}{\pi}\int_0^\infty [a_n x^n f(x) + a_{n-2}x^{n-2} f(x) + \cdots + a_2 x^2 f(x) + a_0 f(x)] \cos x\omega dx$$

$$= \frac{2}{\pi}\int_0^\infty a_0 f(x) \cos x\omega dx + \frac{2}{\pi}\int_0^\infty a_2 x^2 f(x) \cos x\omega dx + \cdots + \frac{2}{\pi}\int_0^\infty a_n x^n f(x) \cos x\omega dx$$

By using theorem 1 and after simplification, we obtain:

$${A_n}^1(\omega) = a_0 A(\omega) - a_2 \frac{d}{d\omega}\big(A(\omega)\big) + \cdots + a_n (-1)^{\left(\frac{n}{2}\right)} \frac{d^n}{d\omega^n}\big(A(\omega)\big)$$

$$= \sum_{m=0}^{n} a_n \, (-1)^{\left(\frac{n}{2}\right)} \frac{d^n}{d\omega^n}\big(A(\omega)\big)$$

Where, $m = 0, 2, 4, 6, \dots, n$

Similarly the Fourier cosine integral of new function $h(x)f(x)$ is given by:

$$h(x)f(x) = \int_0^\infty {A_n}^2(\omega) \cos x\omega d\omega$$

Where,

$${A_n}^2(\omega) = \frac{2}{\pi}\int_0^\infty h(x)f(x) \cos x\omega dx$$

$$= \frac{2}{\pi}\int_0^\infty [a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_1 x] f(x) \cos x\omega dx$$

$$= \frac{2}{\pi} \int_{0}^{\infty} [a_{n-1} x^{n-1} f(x) + a_{n-3} x^{n-3} f(x) + \cdots + a_1 x f(x)] \cos x\omega dx$$

$$= \frac{2}{\pi} \int_{0}^{\infty} a_1 f(x) \cos x\omega dx + \frac{2}{\pi} \int_{0}^{\infty} a_3 x^3 f(x) \cos x\omega dx + \cdots + \frac{2}{\pi} \int_{0}^{\infty} a_{n-1} x^{n-1} f(x) \cos x\omega dx$$

By using theorem 1 and after simplification, we obtain:

$$A_n{}^2(\omega) = a_1 \frac{d}{d\omega}\big(B(\omega)\big) - a_3 \frac{d^3}{d\omega^3}\big(B(\omega)\big) + \cdots + a_{n-1}(-1)^{\left(\frac{n+3}{2}\right)} \frac{d^{n-1}}{d\omega^{n-1}}\big(B(\omega)\big)$$

$$= \sum_{r=0}^{n-1} a_{n-1} (-1)^{\left(\frac{n+3}{2}\right)} \frac{d^{n-1}}{d\omega^{n-1}}\big(B(\omega)\big)$$

Where, $r = 1, 3, 5, 7, \dots, n-1$

Hence, from the combination of $A_n{}^1(\omega)\ and\ A_n{}^2(\omega)$, we obtain:

$$A_n{}^*(\omega) = \sum_{m=0}^{n} a_n (-1)^{\left(\frac{n}{2}\right)} \frac{d^n}{d\omega^n}\big(A(\omega)\big) + \sum_{m=0}^{n-1} a_{n-1} (-1)^{\left(\frac{n+3}{2}\right)} \frac{d^{n-1}}{d\omega^{n-1}}\big(B(\omega)\big)$$

Where, $m = 0, 2, 4, 6, \dots, n \quad and \;\; r = 1, 3, 5, \dots, n-1$

Such that the Fourier cosine integral $P_n(x)f(x)$ is:

$$P_n(x)f(x) = \int_{0}^{\infty} A_n{}^*(\omega) \cos x\omega d\omega$$

B) Consider a polynomial function:

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$$

with real numbers coefficients and assume that $f(x)$ has Fourier sine integral. Then, for any natural number n, the Fourier sine integral of a new function, $P_n(x)f(x)$, given by:

$$P_n(x)f(x) = \int_{0}^{\infty} B_n{}^*(\omega) \sin x\omega d\omega$$

Such that,

$$B_n{}^*(\omega) = \sum_{m=0}^{n} a_n(-1)^{\left(\frac{n}{2}\right)} \frac{d^n}{d\omega^n}\big(B(\omega)\big) + \sum_{r=0}^{n-1} a_{n-1}(-1)^{\left(\frac{n+3}{2}\right)} \frac{d^{n-1}}{d\omega^{n-1}}\big(A(\omega)\big)$$

Where, $m = 0, 2, 4, 6, \dots, n \quad and \ r = 1, 3, 5, \dots, n-1$

Proof

Suppose $f(x)$ has a Fourier sine integral and $P_n(x)$ be any polynomial function of degree n:

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

$$= a_n x^n + +a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0 + a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_1 x$$

$$= g(x) + h(x)$$

Where, $g(x) = a_n x^n + +a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0$ implies all powers of variables are even and $h(x) = a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_1 x$ with all powers of variables are odd.

Then, the Fourier sine integral of a new function $g(x)f(x)$ is given by:

$$g(x)f(x) = \int_0^\infty B_n{}^1(\omega) \sin x\omega d\omega$$

Where,

$$B_n{}^1(\omega) = \frac{2}{\pi}\int_0^\infty g(x)f(x) \sin x\omega dx$$

$$= \frac{2}{\pi}\int_0^\infty [a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0] f(x) \sin x\omega dx$$

$$= \frac{2}{\pi}\int_0^\infty [a_n x^n f(x) + a_{n-2}x^{n-2} f(x) + \cdots + a_2 x^2 f(x) + a_0 f(x)] \sin x\omega dx$$

$$= \frac{2}{\pi}\int_0^\infty a_0 f(x) \sin x\omega dx + \frac{2}{\pi}\int_0^\infty a_2 x^2 f(x) \sin x\omega dx + \cdots + \frac{2}{\pi}\int_0^\infty a_n x^n f(x) \sin x\omega dx$$

By using theorem 1 and after simplification, we obtain:

$$B_n{}^1(\omega) = a_0 B(\omega) - a_2 \frac{d}{d\omega}\big(B(\omega)\big) + \cdots + a_n(-1)^{\left(\frac{n}{2}\right)} \frac{d^n}{d\omega^n}\big(B(\omega)\big)$$

$$= \sum_{m=0}^{n} a_n \,(-1)^{\left(\frac{n}{2}\right)} \frac{d^n}{d\omega^n}\big(B(\omega)\big)$$

Where, $m = 0, 2, 4, 6, \dots, n$

Similarly the Fourier sine integral of new function $h(x)f(x)$ is given by:

$$h(x)f(x) = \int_0^\infty B_n{}^2(\omega) \sin x\omega d\omega$$

Where,

$$B_n{}^2(\omega) = \frac{2}{\pi}\int_0^\infty h(x)f(x) \sin x\omega dx$$

$$= \frac{2}{\pi}\int_0^\infty [a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_1 x] f(x) \sin x\omega dx$$

$$= \frac{2}{\pi}\int_0^\infty [a_{n-1}x^{n-1}f(x) + a_{n-3}x^{n-3}f(x) + \cdots + a_1 x f(x)] \sin x\omega dx$$

$$= \frac{2}{\pi}\int_0^\infty a_1 f(x) \sin x\omega dx + \frac{2}{\pi}\int_0^\infty a_3 x^3 f(x) \sin x\omega dx + \cdots + \frac{2}{\pi}\int_0^\infty a_{n-1}x^{n-1} f(x) \sin x\omega dx$$

By using theorem 1 and after simplification, we obtain:

$$B_n{}^2(\omega) = a_1 \frac{d}{d\omega}\big(A(\omega)\big) - a_3 \frac{d^3}{d\omega^3}\big(A(\omega)\big) + \cdots + a_{n-1}(-1)^{\left(\frac{n+3}{2}\right)} \frac{d^{n-1}}{d\omega^{n-1}}\big(A(\omega)\big)$$

$$= \sum_{r=0}^{n-1} a_{n-1} \,(-1)^{\left(\frac{n+3}{2}\right)} \frac{d^{n-1}}{d\omega^{n-1}}\big(A(\omega)\big)$$

Where, $r = 1, 3, 5, 7, \dots, n-1$

Hence, from the combination of $B_n{}^1(\omega)$ *and* $B_n{}^2(\omega)$, we obtain:

$$B_n{}^*(\omega) = \sum_{m=0}^{n} a_n(-1)^{\left(\frac{n}{2}\right)} \frac{d^n}{d\omega^n}\big(B(\omega)\big) + \sum_{r=0}^{n-1} a_{n-1}(-1)^{\left(\frac{n+3}{2}\right)} \frac{d^{n-1}}{d\omega^{n-1}}\big(A(\omega)\big)$$

Where, $m = 0, 2, 4, 6, \dots, n \quad and \; r = 1, 3, 5, \dots, n-1$

Such that the Fourier sine integral $P_n(x)f(x)$ is:

$$P_n(x)f(x) = \int_0^\infty B_n{}^*(\omega) \sin x\omega d\omega$$

4. Conclusion

The purpose of this paper is to provide a brief representation any function in integral form, Fourier cosine and sine transforms, after multiplying the given function by power functions and polynomial functions and provide the relation between Fourier Cosine transforms and Fourier Sine integrals. It is hoped that this implementation of the Fourier cosine and sine integrals will help representing the solutions of ODEs, PDEs, and integral equations that involving power functions terms in the integral form of simpler functions Cosine and Sine.

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