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Numerical Approximation for Third Order Korteweg-De Vries (KDV) Equation

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Abstract

In this paper, Natural transform and Homotopy perturbation methods are coupled to study third order Korteweg-De Vries (KDV) equation analytically. The introduced technique is useful to obtain closed form solutions. The combined method required less computational effort when compared with some existing methods and reduced volume of calculations. Three illustrative examples are used to demonstrate the effectiveness of the method.

Keywords: Homotopy perturbation method; Korteweg-De Vries Equation; Natural transform method.

1. Introduction

Nonlinear models are useful in the description of phenomena arising in physics, chemistry, engineering and other sciences. However, obtaining exact analytical solutions for these problems, except in limited cases could pose some difficulties [1]. Numerical calculation methods were good means of analyzing the nonlinear equations and their improvement led to improvement in analytical methods. In recent years, combination of numerical and analytical methods have drawn special attention. Example of such is homotopy perturbation method established by He in 1998 to obtain series solution of nonlinear differential equations [2,3]. The method has merits of simplicity and easy execution. Korteweg-De Vries equation is a mathematical model of waves on shallow water surfaces. It is a nonlinear partial differential equation whose solutions can be exactly and precisely specified. It was first introduced by Boussinesq in 1877 and rediscovered by Diederik Korteweg and Gustav de Vries in 1895.

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The third order KDV equation can be presented in the form:

$$v_t + avv_x + bv_{xxx} = 0 \ ; \quad v(x,0) = f(x) \tag{1.1}$$

Where a and b are constant.

Laplace transforms method has been combined with homotopy perturbation method to solve KDV equation [4] and other nonlinear problems [5]. Similarly, Aboodh transform has been combined with homotopy perturbation method to calculate approximate solution of some third order KDV equations with initial conditions [6].

The Natural transform is similar to Laplace Integral transform [7].

The purpose of this paper is to enhance the application of the Natural transform method, by coupling it with the homotopy perturbation method known as Natural homotopy perturbation method (NHPM) for the solution of Korteweg-De Vries (KDV) Equation. The method has been successfully applied for obtaining exact solutions of Linear and Nonlinear Schrodinger Equations [8] and time dependent functional differential equations [9].

2. Basic Idea

We illustrate the basic idea of NHPM by considering a general form of nonlinear non homogenous partial differential equation as follows:

$$Dv(x,t) + Rv(x,t) + Nv(x,t) = g(x,t) \tag{2.1}$$

With the following initial conditions

$$v(x,0) = h(x) \ ; \quad v_t(x,0) = f(x)$$

$D = \dfrac{\partial}{\partial t^2}$, R

where D is the second order linear differential operatoris the linear differential operator of less order than D, N represents the general nonlinear differential operator and $g(x,t)$ is the source term.

Applying the Natural transform to equation (2.1) subject to the given initial condition, we have

$$N^{+}[Dv(x,t)] + N^{+}[Rv(x,t)] + N^{+}[Nv(x,t)] = N^{+}[g(x,t)] \tag{2.2}$$

using differentiation property of Natural transform and above initial conditions, we have

$$V(x,s,v) = \frac{1}{s}h(x) + \frac{v}{s^2}f(x) - \frac{v^2}{s^2}\mathrm{N}^{+}[RV(x,t)] - \frac{v^2}{s^2}\mathrm{N}^{+}[NV(x,t)] + \frac{v^2}{s^2}\mathrm{N}^{+}[g(x,t)] \tag{2.3}$$

Operating with the inverse Natural transform on both sides of equation (2.3), we have

$$v(x,t)=G(x,t)-\mathrm{N}^{-1}\left[\frac{v^2}{s^2}\mathrm{N}^{+}\left[Rv(x,t)+\mathrm{N}\,v(x,t)\right]\right] \tag{2.4}$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial condition.

Now, applying the homotopy perturbation method (HPM)

$$v(x,t)=\sum_{n=0}^{\infty}P^n v_n(x,t) \tag{2.5}$$

and the nonlinear term can be decomposed as

$$Nv(x,t)=\sum_{n=0}^{\infty}P^n H_n(v) \tag{2.6}$$

where $H_n(v)$ are He's polynomials and can be evaluated using the following formula[7]:

$$H_n(v_1,v_2,\ldots,v_n)=\frac{1}{n!}\frac{\partial^n}{\partial P^n}\left[N\sum_{j=0}^{n}P^J V_J\right]_{p=0}\;;\; n=0,1,2,\ldots \tag{2.7}$$

Substituting equations (2.5) and (2.6) in (2.4); we have

$$\sum_{n=0}^{\infty}P^n v_n(x,t)=G(x,t)-P\left(\mathrm{N}^{-1}\left[\frac{v^2}{s^2}\mathrm{N}^{+}\left[R\sum_{n=0}^{\infty}P^n v_n(x,t)+\sum_{n=0}^{\infty}P^n H_n(v)\right]\right]\right) \tag{2.8}$$

which is the coupling of the Natural transform and the homotopy perturbation method using He's polynomials.

Comparing the coefficient of same powers of P, we obtain the following approximations:

$$\begin{aligned}
P^0 &: v_0(x,t)=G(x,t)\\
P^1 &: v_1(x,t)=-\mathrm{N}^{-1}\left[\frac{v^2}{s^2}\mathrm{N}^{+}\left[Rv_0(x,t)+H_0(v)\right]\right]\\
P^2 &: v_2(x,t)=-\mathrm{N}^{-1}\left[\frac{v^2}{s^2}\mathrm{N}^{+}\left[Rv_1(x,t)+H_1(v)\right]\right]\\
P^3 &: v_3(x,t)=-\mathrm{N}^{-1}\left[\frac{v^2}{s^2}\mathrm{N}^{+}\left[Rv_2(x,t)\right]+H_2(v)\right]
\end{aligned} \tag{2.9}$$

and so on.

Thus, the series solution of equation (2.1) is

$$v(x,t) = \lim_{k \to \infty} \sum_{n=0}^{k} v_n(x,t) \qquad (2.10)$$

3. Applications

Here, the effectiveness and usefulness of Natural Homotopy Perturbation Method (NHPM) are demonstrated by finding exact solutions of three Korteweg-De Vries (KDV) Equations.

Example 3.1 Consider the following linear homogenous KDV equation

$$v_t - 6vv_x + v_{xxx} = 0 \qquad (3.1)$$

with initial condition $v(x,0) = 6x \qquad (3.2)$

applying the natural homotopy perturbation method on both sides of equation (3.1) subject to the initial condition (3.2) , we have

$$v(x,s) = \frac{6x}{s} + \frac{v}{s}\mathrm{N}^{+}\left[6vv_x - v_{xxx}\right] \qquad (3.3)$$

The inverse of natural transform implies that

$$v(x,t) = 6x + \mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[6vv_x - v_{xxx}\right]\right] \qquad (3.4)$$

Now, we apply the homotopy perturbation method to get

$$\sum_{n=0}^{\infty} P^n v_n(x,t) = 6x + P\left(\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[6\sum_{n=0}^{\infty} P^n H_n(v) - \sum_{n=0}^{\infty} P^n v_{nxxx}\right]\right]\right) \qquad (3.5)$$

Where $H_n(v)$ are He's polynomials that represents the nonlinear terms.

The first few components of He's polynomials are given by:

$$\begin{aligned} H_0(v) &= v_0 v_{0x} \\ H_1(v) &= v_0 v_{1x} + v_1 v_{0x} \\ H_2(v) &= v_0 v_{2x} + v_1 v_{1x} + v_2 v_{0x} \end{aligned} \qquad (3.6)$$

Comparing the coefficients of like powers of P in eqn (3.5); we obtain the following approximations.

$$P^0: \quad v_0(x,t) = 6x$$

$$P^1: \quad v_1(x,t) = \mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[6H_0(v) - v_{0xxx}\right]\right] = \mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[6v_0 v_{0x} - v_{0xxx}\right]\right] = 6^3 xt$$

$$P^2: \quad v_2(x,t) = \mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[6H_1(v) - v_{1xxx}\right]\right] = \mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[6(v_0 v_{1x} + v_1 v_{0x}) - v_{1xxx}\right]\right] = 6^5 xt^2$$

$$P^3: \quad v_3(x,t) = \mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[6H_2(v) - v_{2xxx}\right]\right] = \mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[6(v_0 v_{2x} + v_1 v_{1x} + v_2 v_{0x}) - V_{2xxx}\right]\right] = 6^7 xt^3$$

and so on.

Therefore, the solution $v(x,t)$ is given by

$$v(x,t) = 6x\left(1 + 36t + (36t)^2 + (36t)^3 + \quad . \ . \ .\right) \qquad (3.7)$$

Closed form solution of (3.1) is $v(x,t) = \dfrac{6x}{1-36t}$, $|36t| < 1$ $\qquad (3.8)$

which is similar to results obtained using Homotopy perturbation Transform method HPTM [4], Aboodh transform method [6] and Variation iterative method VIM [10].

Example 3.2: Consider the following homogenous KDV equation

$$v_t + v v_x + v_{xxx} = 0 \qquad (3.9)$$

with initial condition $v(x,0) = 1 - x$ $\qquad (3.10)$

Applying the natural transform on both sides of equation (3.9) subject to the initial condition (3.10), we have

$$v(x,s) = \frac{1-x}{s} - \frac{v}{s}\mathrm{N}^{+}\left[v v_x + v_{xxx}\right] \qquad (3.11)$$

The inverse of Natural transform implies that:

$$v(x,t) = (1-x) - \mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[vv_x + v_{xxx}\right]\right] \qquad (3.12)$$

Now, we apply the homotopy perturbation method to get:

$$\sum_{n=0}^{\infty} P^n v_n(x,t) = (1-x) - P\left(\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[\sum_{n=0}^{\infty} P^n H_n(v) + \sum_{n=0}^{\infty} P^n v_{nxxx}\right]\right]\right) \qquad (3.13)$$

Comparing the coefficients of like powers of P in eqn (3.13), we obtain the following approximations:

$$P^0 : v_0(x,t) = 1-x$$
$$P^1 : v_1(x,t) = -\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[H_0(v) + v_{0xxx}\right]\right] = -\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[v_0 v_{0x} + v_{0xxx}\right]\right] = (1-x)t$$
$$P^2 : v_2(x,t) = -N^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[H_1(v) + v_{1xxx}\right]\right] = -\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[v_1 v_{1x} + v_1 v_{0x} + v_{1xxx}\right]\right] = (1-x)t^2$$
$$P^3 : v_3(x,t) = -N^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[H_2(v) + v_{2xxx}\right]\right] = -\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[v_0 v_{2x} + v_1 v_{1x} + v_2 v_{0x} + v_{2xxx}\right]\right] = (1-x)t^3$$

and so on.

Therefore, the solution $v(x,t)$ is given by

$$v(x,t) = (1-x)\left(1 + t + t^2 + t^3 + \ . \ . \ .\right) \qquad (3.14)$$

Closed form solution of (3.9) is $v(x,t) = \dfrac{1-x}{1-t}, \quad |t| < 1 \qquad (3.15)$

Which is similar to results obtained using Aboodh transform [6] and Variation iterative method [10].

Example 3.3: Consider the following homogenous KDV equation

$$v_t - 6vv_x + v_{xxx} = 0 \qquad (3.16)$$

with initial condition $\qquad v(x,0) = \dfrac{2}{(x-3)^2} \qquad (3.17)$

Applying the natural transform on both sides of eqn (3.16) subject to the initial condition (3.17) we have

$$v(x,s)=\frac{1}{s}\cdot\frac{2}{(x-3)^2}+\frac{v}{s}\mathrm{N}^{+}\left[6vv_x-v_{xxx}\right] \qquad (3.18)$$

The inverse of Natural transform implies that

$$v(x,t)=\frac{2}{(x-3)^2}+\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[6vv_x-v_{xxx}\right]\right] \qquad (3.19)$$

Now, we apply the homotopy perturbation method to get:

$$\sum_{n=0}^{\infty}P^n v_n(x,t)=\frac{2}{(x-3)^2}+P\left(N^{-1}\left[\frac{v}{s}N^{+}\left(6\sum_{n=0}^{\infty}P^n H_n(v)-\sum_{n=0}^{\infty}P^n v_{nxxx}\right)\right]\right) \qquad (3.20)$$

Comparing the coefficients of like powers of *P* in eqn (3.20), we obtain the following approximations

$$P^0:v_0(x,t)=\frac{2}{(x-3)^2}$$

$$P^1:v_1(x,t)=\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[6H_0(v)-v_{0xxx}\right]\right]=\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[6v_0v_{0x}-v_{0xxx}\right]\right]=0$$

$$P^2:v_2(x,t)=\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[6H_1(v)-v_{1xxx}\right]\right]=\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[6(v_0v_{1x}+v_1v_{0x})-v_{1xxx}\right]\right]=0$$

$$P^3:v_3(x,t)=\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[6H_2(v)-v_{2xxx}\right]\right]=\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[6(v_0v_{2x}+v_1v_{1x}+v_2v_{0x})-v_{2xxx}\right]\right]=0$$

and so on.

Therefore, the solution $v(x,t)$ is given by

$$v(x,t)=\frac{2}{(x-3)^2} \qquad (3.21)$$

Which is the closed form solution and similar to result obtained using variation iterative method (VIM) [10].

4. Conclusion

In this paper, we have applied the natural homotopy perturbation method (NHPM) developed by Maitama and his colleagues [8] to solve third order Korteweg-De Vries (KDV) equation.

The method leads to exact solution of the equation using the initial condition. The recurrent relations obtained show the effectiveness of the method in the solution of nonlinear partial differential equations of this type with wide applications in Applied Mathematics and Engineering.

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