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Convergence Analysis of Finite Difference Method for Differential Equation

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Abstract

In this paper, convergence analysis of a finite difference method for the linear second order boundary value ordinary differential equation is determined by investigating basic key concepts such as *consistency* and *stability* by using the maximum norm.

Keywords: Finite difference method; Differential equation; Error; stability; Consistency.

1. Introduction

A differential equation involving derivatives with respect to single independent variable is called an ordinary differential equation (ODE) [1]. An ODE is known as linear if the derivative of the dependent variable is one and also the power of the dependent variable is one and the coefficient of the dependent variable are constants or independent variables [2]. A differential equation to be satisfied over a region together with a set of boundary conditions is said to be boundary value differential equation. Boundary value problems occur very frequently in various fields of science and engineering such as mechanics, quantum physics, electro hydro dynamics, and theory of thermal expansions [3]. There are different ways of arriving at different approximations for the solution of differential equations. The best method is the one which gives best approximation for the solution, i.e. which have minimum error.

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2. Finite Difference Approximations

Methods involving finite differences for solving BVPs replace each of the derivatives in the differential equation with an appropriate difference-quotient approximation [4]. We shall consider the linear two-point ordinary boundary value problem (BVP) of the form

$$y"(x) + p(x)y' + q(x)y = r(x), y(a) = y_0, y(b) = y_n \qquad (1.1)$$

satisfies the following conditions to assure the existence of unique solution, $p(x), q(x)$, and $r(x)$ are continuous on $[a, b]$, and $q(x) < 0$ on $[a, b]$(for positive $q(x)$ the BVP may not possess a solution [5]). For the sake of convenience, we shall employ equal increments in the independent variable. Then $x_0, x_1, \dots, x_n$ are the interior mesh points of the interval $[a, b]$ related as $x_i = x_0 + ih$ for $i = 0, 1, \dots, n$ and h is the step size with $h = (b - a)/n$. Methods involving finite differences for solving boundary-value problems replace each of the derivatives in the differential equation by an appropriate difference-quotient approximation [6]. The particular difference quotient and step size h are chosen to maintain a specified order of truncation error. However, h cannot be too small becomes of the general instability of the derivative approximation [4]. If $y \epsilon \mathbb{C}^4[a, b]$, then by replacing the derivatives in (1.1) by the following central differences which are derived from Taylor's theorem can be obtained as follows.

$$y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6} y^{(3)}(\eta_i) y''(x_i) = \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{12} y^{(4)}(\xi_i)$$

for some η_i and ξ_i in the interval (x_{i-1}, x_{i+1}).

Let y_i denote the numerical approximate and $y(x_i)$ the exact(analytical) values for (1.1) respectively. Then with truncation error, the approximate difference equations becomes

$y'_i \approx \frac{y_{i+1} - y_{i-1}}{2h}$ and $y''_i \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$

Substituting these in equation (1.1) with $p(x_i) = p_i$, $q(x_i) = q_i$ and $r(x_i) = r_i$ the BVP becomes

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + p_i \frac{y_{i+1} - y_{i-1}}{2h} + q_i y_i = r_i,$$

$$i = 1, 2, \dots, n - 1 \qquad (1.2)$$

Multiplying the difference equation (1.2) by h^2 and rearranging the result gives the following equation

$$\left(1 + \frac{h}{2} p_i\right) y_{i-1} + (2 + h^2 q_i) y_i - \left(1 - \frac{h}{2} p_i\right) y_{i+1} = h^2 r_i \qquad (1.3)$$

This is a finite difference equation which is an approximation to the differential equation (1.1) at the interior mesh point $x_1, x_2, \dots, x_{n-1}$ of the interval $[a, b]$. By replacing $i = 1, 2, \dots, n - 1$ in (1.3), this gives $n - 1$ linear equations with the unknowns $y_1, y_2, \dots, y_{n-1}$ which can be solved using Gaussian elimination method with back substitution.

3. Convergence Analysis of the Method

To be equation (1.3) a convergent solution for (1.1), we need to estimate the maximum error for the appropriate selection of h.

Without truncating the error term, equation (1.2) becomes

$$\frac{y(x_{i+1})-2y(x_i)+y(x_{i-1})}{h^2}-\frac{h^2}{12}y^{(4)}(\xi_i)+p_i\frac{y(x_{i+1})-y(x_{i-1})}{2h}-\frac{h^2}{6}y^{(3)}(\eta_i)+q_iy(x_i)=r(x_i) \qquad (1.4)$$

As it is stated in [6], if we subtract (1.2) from (1.4) and using $e_i = y(x_i) - y_i$, the result is

$$\frac{e_{i+1}-2e_i+e_{i-1}}{h^2}+p_i\frac{e_{i+1}-e_{i-1}}{2h}+q_ie_i=h^2g_i$$

Where e_i is the global error and

$$g_i=\frac{1}{12}y^{(4)}(\xi_i)+\frac{1}{6}y^{(3)}(\eta_i)$$

From this result, one can observe that h^2g_i is the local truncation error of the method. As the value of h close to 0, the truncation error vanishes and hence the finite difference method (1.3) becomes *consistent*.

After collecting like terms and multiplying both sides of (1.5) by h^2 gives the following equation

$$\left(1+\frac{h}{2}p_i\right)e_{i-1}+(2+h^2q_i)e_i-\left(1-\frac{h}{2}p_i\right)e_{i+1}=h^4g_i \qquad (1.6)$$

To measure the magnitude of this vector we must use some *norm*, for instance the max-norm because it is used to measure grid functions and it is easy to bound.

$$\|e\|_\infty=\max_{1\le i\le n}|e_i|=\max_{1\le i\le n}|y(x_i)-y_i|$$

$$\Rightarrow (2+h^2q_i)e_i=\left(1-\frac{h}{2}p_i\right)e_{i+1}-\left(1+\frac{h}{2}p_i\right)e_{i-1}+h^4g_i$$

$$\Rightarrow |2+h^2q_i||e_i|\le\left|1-\frac{h}{2}p_i\right||e_{i+1}|+\left|1+\frac{h}{2}p_i\right||e_{i-1}|+h^4|g_i|$$

$$\Rightarrow |2+h^2q_i|\|e\|_\infty\le\left|1-\frac{h}{2}p_i\right|\|e\|_\infty+\left|1+\frac{h}{2}p_i\right|\|e\|_\infty+h^4\|g\|_\infty$$

$$\Rightarrow |2+h^2q_i|\|e\|_\infty\le 2\|e\|_\infty+h^4\|g\|_\infty$$

$\Rightarrow h^2|q_i|\|e\|_\infty \leq h^4\|g\|_\infty$

$\Rightarrow \|e\|_\infty \leq \dfrac{h^2\|g\|_\infty}{q_i}$

Hence the upper bound for $\|e\|_\infty$ is

$$\|e\|_\infty \leq \frac{h^2\|g\|_\infty}{\inf|q(x_i)|} \tag{1.7}$$

This is just the largest error over the interval. If this error (1.7) converges to zero without having $h^2\|g\|_\infty$ converging to 0, the solution method (1.3) becomes *stable*. **Example**: Consider the following BVP

$y''(x) = \frac{2x}{1+x^2}y'(x) - \frac{2}{1+x^2}y(x) + 1,$

with $y(0) = 1.25$ and $y(4) = -0.95$ over the interval [0,4]. Both the exact and the numerical solution with step size $h = 0.2$ is shown on the table below correct to 5 decimal places as it is stated in [7] and [8] with some modification.

Table 1: Numerical approximation and exact solution for the differential equation

x_i	y_i	Exact value	Error(e)
0.2	1.314503	1.317350	0.002847
0.4	1.320607	1.326505	0.005898
0.6	1.272755	1.281762	0.009007
0.8	1.177399	1.189412	0.012013
1.0	1.042106	1.056886	0.014780
1.2	0.874878	0.892086	0.017208
1.4	0.683712	0.702947	0.019235
1.6	0.476372	0.497187	0.020815
1.8	0.260264	0.282184	0.021920
2.0	0.042399	0.064931	0.022533
2.2	-0.170616	-0.147977	0.022639
2.4	-0.372557	-0.350325	0.022232
2.6	-0.557565	-0.536261	0.021304
2.8	-0.720114	-0.700262	0.019852
3.0	-0.854988	-0.837116	0.017872
3.2	-0.957250	-0.941888	0.015362
3.4	-1.022221	-1.009899	0.012322
3.6	-1.045457	-1.036709	0.008749
3.8	-1.022727	-1.018086	0.004641

$$y''(x) = \frac{2x}{1+x^2}y'(x) - \frac{2}{1+x^2}y(x) + 1$$

From equation (1.7) we have

$$h^2\|g\|_\infty \geq \|e\|_\infty \min|q(x_i)| \tag{1.8}$$

From (1.8), the maximum value of local truncation error is related with $\|e\|_\infty \min|q(x_i)|$.

But from the result on table 1, the maximum error is

$$\|e\|_\infty = 0.022639$$

$$\min|q(x_i)| = \frac{2}{1+x^2{}_{max}} = \frac{2}{1+(3.8)^2}$$

$$= 0.12953368$$

$$\Longrightarrow \|e\|_\infty \min|q(x_i)| = 0.002932513$$

This is closer to maximum truncation error. The remaining error

i.e. $0.022639 - 0.002932513 = 0.01933139$

must be the error occurred by instability of the method.

Graphically, the exact solution and the finite difference approximation is shown below.

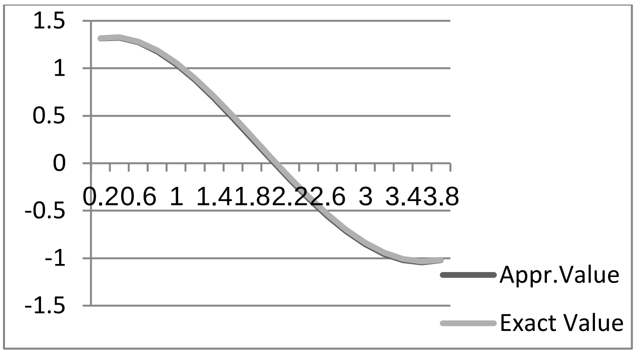

Figure 1: Numerical approximation and exact solutions for the differential equation

$$y''(x) = \frac{2x}{1+x^2}y'(x) - \frac{2}{1+x^2}y(x) + 1$$

4. Conclusion

For the appropriate selection of step size h and for the BVP of the form (1.1) having large magnitude for the minimum value of $q(x)$, the finite difference method becomes both consistent and stable hence the finite difference method (1.3) becomes *convergent.*

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