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On Some Characteristics of a Simple Random Walk

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Abstract

In most of the probabilistic problems Markov chain models are used and the random walk models are one of the most essential class of the Markov chains. A random walk model appears in many real world problems such as a gambling problem, the motion of a certain particle, the price change in a stock exchange market and the real-time change in a network traffic. In this present study a simple random walk $\{\mathrm{X_n}\}$ is defined and probability distribution function is obtained. After that, the mean, the second moment and the variance of this simple walk is obtained. Also the autocorrelation function $\mathrm{R_X}$ is given. Furthermore the mean and variance of the increment $\mathrm{X_m} - \mathrm{X_n}$ are calculated.

Keywords: Markov chain; random walk; simple random walk; variance; autocorrelation function; covariance.

1. Introduction

In most of the probabilistic problems Markov chain models are used and the random walk models are one of the most essential class of the Markov chains. A random walk model appears in many real world problems such as a gambling problem, the motion of a certain particle, the price change in a stock exchange market and the real-time change in a network traffic. In [1], random walks on integers is studied. A simple random walk $S_n = X_1 + X_2 + \cdots + X_n;\ n \geq 1;\ S_0 = 0$ and the first time (N) that this random walk visits the state 1 is given by [2].

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A brief study on random walk processes is given in [3]. Upper and lower bounds on the speed of a one dimensional excited random walk is studied by [4]. In [5] random walk in a high density dynamic random environment is studied. In this present study a simple random walk $\{X_n\}$ is defined and probability distribution function is obtained. After that, the mean, the second moment and the variance of this simple walk is obtained. Also the autocorrelation function R_X is given. Furthermore the mean and variance of the increment $X_m - X_n$ are calculated.

2. Random Walk

Suppose that we make a one-dimensional random walk on the real line. We start at a given initial position X_0 on the x −axis at time $t = 0$. At time $t = 1$ we jump to position X_1.So that the step size $S_1 = X_1 - X_0$ is a random variable with some distribution $F(s)$. By time $t = 2$ we jump by another amount S_2, that S_2 is independent of S_1 but has the same distribution $F(s)$. Proceeding this way, our position after n jumps, or at time $t = n$, is thus given by as following

$$X_n = X_0 + S_1 + S_2 + \cdots + S_n \tag{1}$$

where $\{S_i\}$ is a set of independently and identically distributed random variables with a common distribution $F(s)$. The discrete time sequence $\{X_n\}$ is called a one-dimensional random walk [7].

2.1. The Simple Random Walk

A simple random walk is defined as a special case of the random walk model, in which only two values are possible for each step S_i, either $+1$ or -1. Thus the position at time $t = n$ is:

$$X_n = X_0 + \sum_{i=1}^{n} S_i\,, \qquad n = 1, 2, 3, \ldots \tag{2}$$

The simple random walk $\{X_n\}$ has the following properties [8]:

1. Spatial homogeneity:

$$P(X_n = k|X_0 = a) = P(X_n = k + b|X_0 = a + b) \tag{3}$$

that is, the distribution of $X_n - X_0$ does not depend on the initial value of X_0

2. Temporal homogeneity:

$$P(X_n = k|X_0 = a) = P(X_{n+m} = k|X_m = a) \tag{4}$$

that is, $X_{n+m} - X_m$ has the same distribution as $X_n - X_0$ for all $m, n \geq 0$

3. Independent increments: For a set of disjoint intervals $(m_i, n_i], i = 1, 2, ...$ the increments $(X_{n_i} - X_{m_i})$ are independent.

4. Markov property: The sequence $\{X_n\}$ is a simple Markov chain:

$$P(X_{n+m} = k|X_0, X_1, \dots, X_n) = P(X_{n+m} = k|X_n), \qquad m \geq 0 \tag{5}$$

2.1.1. Obtaining the mean, second moment and variance of simple random walk

Because the simple random walk is spatially homogeneous (first property given by equation 3), let us assume

$$X_0 = a = 0.$$

Suppose that out of n random steps, n_1 steps are taken to the right $(+1)$ and n_2 steps are to the left (-1). It is obvious that these steps are independent. Now let us assume the following probabilities:

$$S_i = \begin{cases} +1 & , \quad \text{with probability p} \\ -1 & , \quad \text{with probability } q = 1 - p \end{cases} \tag{6}$$

Let the position after n steps be $X_n = n_1 - n_2 \equiv k$. Since $n_1 + n_2 = n$ we have $n_1 = (n + k)/2$ and $n_1 = (n - k)/2$. Hence,

$$P(X_n = k) = \binom{n}{\frac{n+k}{2}} p^{(n+k)/2} q^{(n-k)/2} \;, \quad k = -n, -n+2, \dots, n-2, n \tag{7}$$

Notice that both $(n + k)$ and $(n - k)$ are even. So that for any state k, $P(X_n = k) = 0$ for all n such that $(n + k)$ is odd. Hence, $\{X_n\}$ is a Markov chain with period $d = 2$.

The mean, second moment and variance of X_n are calculated respectively, as following

$$E(X_n) = \sum_{i=1}^{n} E(S_i) = nE(S_i) = n(p - q) \tag{8}$$

$$E(X_n{}^2) = \sum_{i=1}^{n} E(S_i{}^2) + \sum_{i \neq j} \sum_{j} E(S_i)\, E(S_j) = n + (n^2 - n)(p - q)^2 \tag{9}$$

$$Var(X_n) = E(X_n{}^2) - [E(X_n)]^2 = 4pqn \tag{10}$$

As stated in property 3, the simple random walk is a process with independent increments. The mean of the increment $X_m - X_n$ is :

$$E[X_m - X_n] = (m - n)(p - q) \tag{11}$$

On the other hand, the autocorrelation function $R_X(m,n) = E(X_m X_n),\ m \geq n$ is obtained as:

$$R_X(m,n) = E[(X_m - X_n + X_n)X_n] = E[X_m - X_n]E[X_n] + E\left[{X_n}^2\right]$$

$$= (m-n)n(p-q)^2 + n^2(p-q)^2 + 4pqn$$

$$= mn(p-q)^2 + 4pqn, \qquad m \geq n \tag{12}$$

Since the autocorrelation function is symmetric we have, whether $m \geq n$ or not,

$$R_X(m,n) = mn(p-q)^2 + 4pqmin[m,n] \tag{13}$$

Thus, finally we obtain the covariance between X_m and X_n as:

$$Cov(X_m, X_n) = R_X(m,n) - E[X_m]E[X_n] = 4pqmin[m,n] \tag{14}$$

and the variance of the increment as:

$$Var[X_m - X_n] = Var[X_m] + Var[X_n] - 2Cov(X_m, X_n)$$

$$= 4pq(m+n-2n) = 4pq(m-n), \qquad m \geq n. \tag{15}$$

3. Conclusion and Discussion

In this present study a simple random walk $\{X_n\}$ is defined and probability distribution function is obtained. After that, the mean, the second moment and the variance of this simple walk is obtained. Also the autocorrelation function R_X is given. Furthermore the mean and variance of the increment $X_m - X_n$ are calculated. For further studies probabilistic characteristics of various simple random walk problems can be obtained.

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