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Properties of Fourier Cosine and Sine Transforms

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Abstract

The time and frequency domains are alternative ways of representing signals. The Fourier transform is the mathematical relationship between these two representations. These transformations are of interest mainly as tools for solving ODEs, PDEs and integral equations, and they often also help in handling and applying special functions [8,9]. In this article, I have outlined the main features of properties of Fourier cosine and sine Transforms. These properties demand the implementation of representation of a function in integral form, known as Fourier cosine and sine transforms. The purpose of this paper is to provide a brief representation any function in integral form, Fourier cosine and sine transforms, after multiplying the given function by power functions; $x, x^2, x^3, \ldots x^n$ and provide the relation between Fourier Cosine transforms and Fourier Sine transforms.

Keywords: Fourier transforms; Fourier cosine and sine transforms.

1. Introduction

On 21 December 1807, in one of the most memorable sessions of the French Academy, Jean Baptiste Joseph Fourier, a 21-year old Mathematician and engineer announced a thesis which began a new chapter in the history of Mathematics. Fourier claimed that an arbitrary function, defined in a finite interval by an arbitrary and capricious graph, can always be resolved into a sum of pure sine and cosine. Fourier series in the work of Euler and D. Bernoulli on vibrating strings, but the theory of Fourier series truly began with the profound work of Fourier on heat conduction at the beginning of the 19th century.

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He wanted to use this form to come up with solution to certain linear partial differential equations (specially the heat equation) because sines and cosines behave nicely under differentiation. For instance, he said the derivative of the above functions should be this sum. Fourier deals with the problem of describing the evolution of the temperature $\boldsymbol{T}\ (\boldsymbol{x}, \boldsymbol{t})$ of a thin wire of length $\boldsymbol{\pi}$, stretched between $\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{x} = \boldsymbol{\pi}$, with a constant zero temperature at the ends: $\boldsymbol{T}\ (\boldsymbol{0}, \boldsymbol{t}) = \boldsymbol{0}$ *and* $\boldsymbol{T}\ (\boldsymbol{\pi}, \boldsymbol{t}) = \boldsymbol{0}$. He proposed that the initial temperature $\boldsymbol{T}\ (\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{f}(\boldsymbol{x})$ could be expanded in a series of sine functions. 1828, Dirichlet formulated conditions for a function f(x) to have the Fourier transform f(x) must be single valued have a finite number of discontinuities in any given interval have a finite number of extrema in any given interval be square-integrable [1,2,7].

1.1. Definition

The Fourier cosine transform of the function $f\ (x\ )$ is given by

$$\mathcal{F}_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos x\omega dx$$

Where, c suggests “cosine”. [3], [4], [5]. [6]:

1.2. Definition

The Fourier sine transform of the function $f(x)$ is given by:

$$\mathcal{F}_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin x\omega dx$$

Where, s suggests “sine” [3,4,5,6]:

1.3. Definition

The Fourier transform of the function $f(x)$ is given by:

$$\mathcal{F}\big(f(x)\big) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x) e^{-ix\omega}\, dx$$

References [3,4,5,6].

2. Properties of Fourier Cosine and Sine Transforms

Theorem: (Properties of Cosine and Sine Transforms)

A) Suppose $f(x)$ has a Fourier cosine transforms:

$$\mathcal{F}_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos x\omega dx$$

Then, if $f(x)$ is multiplied by x then, the Fourier cosine transform of a new function $xf(x)$ is

$$\mathcal{F}_c^*\big(f(x)\big) = \frac{d}{d\omega}\left[\mathcal{F}_s\big(f(x)\big)\right]$$

Where, $\mathcal{F}_c^*$ is the Fourier cosine transform of $xf(x)$.

If $f(x)$ is multiplied by x^2 then, the Fourier cosine transform of a new function $x^2f(x)$ is

$$\mathcal{F}_c^{**}\big(f(x)\big) = -\frac{d^2}{d\omega^2}\left[\mathcal{F}_c\big(f(x)\big)\right]$$

Where, $\mathcal{F}_c^{**}$ is the Fourier cosine transform of $x^2f(x)$.

❖ In general,

For n even and if $f(x)$ is multiplied by x^n, then the Fourier cosine transform of $x^nf(x)$ is:

$$\mathcal{F}_c\big(x^nf(x)\big) = (-1)^m\frac{d^n}{d\omega^n}[\mathcal{F}_cf(x)]$$

Where, $m = 1,2,3,\ldots$

For n odd and if $f(x)$ is multiplied by x^n, then the Fourier cosine transform of $x^nf(x)$ is:

$$\mathcal{F}_c\big(x^nf(x)\big) = (-1)^{m+1}\frac{d^n}{d\omega^n}[\mathcal{F}_sf(x)]$$

Where, $m = 1,2,3,\ldots$

Proof

Suppose $f(x)$ has a Fourier cosine and sine transforms; such that

$$\mathcal{F}_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos x\omega dx$$

And

$$\mathcal{F}_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin x\omega dx$$

Let, multiply $f(x)$ by x, then by definition; the Fourier cosine transforms of $xf(x)$ is:

$$\mathcal{F}_c(xf(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty xf(x)\cos x\omega dx$$

Let, $\mathcal{F}_c^* = \mathcal{F}_c(xf(x))$, then from the Fourier sine transform we have:

$$\mathcal{F}_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin x\omega dx$$

by taking differentiation to both sides with respect to ω, we have

$$\frac{d}{d\omega}[\mathcal{F}_s(f(x))] = \frac{d}{d\omega}[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin x\omega dx]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d}{d\omega}[f(x)\sin x\omega]dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\frac{d}{d\omega}[\sin x\omega]dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty xf(x)\cos x\omega dx$$

$$= \mathcal{F}_c(xf(x))$$

$$= \mathcal{F}_c^*$$

Therefore,

$$\mathcal{F}_c^*\big(f(x)\big) = \frac{d}{d\omega}\big[\mathcal{F}_s\big(f(x)\big)\big]$$

Let, multiply $f(x)$ by x^2, then by definition: The Fourier cosine transforms of $x^2f(x))$ is

$$\mathcal{F}_c(x^2 f(x))) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^2 f(x)) \cos x\omega dx$$

Then, from the Fourier cosine transform we have:

$$\mathcal{F}_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos x\omega dx$$

by taking differentiation to both sides with respect to ω twice, we have:

$$\frac{d}{d\omega}\left[\mathcal{F}_c\big(f(x)\big)\right] = \frac{d}{d\omega}[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos x\omega dx]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d}{d\omega}[f(x) \cos x\omega] dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \frac{d}{d\omega}[\cos x\omega] dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty -x f(x) \sin x\omega dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty x f(x) \sin x\omega dx$$

And

$$\frac{d^2}{d\omega^2}\left[\mathcal{F}_c\big(f(x)\big)\right] = \frac{d^2}{d\omega^2}[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos x\omega dx]$$

$$= \frac{d}{d\omega}[\frac{d}{d\omega}(\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos x\omega dx)]$$

$$= \frac{d}{d\omega}[-\sqrt{\frac{2}{\pi}} \int_0^\infty x f(x) \sin x\omega dx]$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d}{d\omega}[xf(x)\sin x\omega]dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty x^2 f(x)\cos x\omega dx$$

$$= -\mathcal{F}_c(x^2 f(x))$$

$$= -\mathcal{F}_c^{**}$$

Therefore, $\mathcal{F}_c^{**} = -\frac{d^2}{d\omega^2}\left[\mathcal{F}_c\left(f(x)\right)\right]$

Continuing the process;

For n even and if $f(x)$ is multiplied by x^n, then the Fourier cosine transform of $x^n f(x)$ is:

$$\mathcal{F}_c\left(x^n f(x)\right) = (-1)^m \frac{d^n}{d\omega^n}[\mathcal{F}_c f(x)]$$

Where, $m = 1,2,3, \ldots$

For n odd and if $f(x)$ is multiplied by x^n, then the Fourier cosine transform of $x^n f(x)$ is:

$$\mathcal{F}_c\left(x^n f(x)\right) = (-1)^m \frac{d^n}{d\omega^n}[\mathcal{F}_s f(x)]$$

Where, $m = 1,2,3, \ldots$

B) Suppose $f(x)$ has Fourier sine transforms:

$$\mathcal{F}_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin x\omega dx$$

Then, if $f(x)$ is multiplied by x then, the Fourier cosine transform of a new function $xf(x)$ is

$$\mathcal{F}_s^*\left(f(x)\right) = -\frac{d}{d\omega}\left[\mathcal{F}_c\left(f(x)\right)\right]$$

Where, $\mathcal{F}_s^*$ is the Fourier sine transform of $xf(x)$.

If $f(x)$ is multiplied by x^2 then, the Fourier sine transform of a new function $x^2 f(x)$ is

$$\mathcal{F}_s^{**}\big(f(x)\big) = -\frac{d^2}{d\omega^2}\big[\mathcal{F}_s\big(f(x)\big)\big]$$

Where, $\mathcal{F}_s^{**}$ is the Fourier cosine transform of $x^2 f(x)$.

❖ In general,

For n even and if $f(x)$ is multiplied by x^n, then the Fourier sine transform of $x^n f(x)$ is:

$$\mathcal{F}_s\big(x^n f(x)\big) = (-1)^{m+1}\frac{d^n}{d\omega^n}[\mathcal{F}_s f(x)]$$

Where, $m = 1,2,3, \ldots$

For n odd and if $f(x)$ is multiplied by x^n, then the Fourier sine transform of $x^n f(x)$ is:

$$\mathcal{F}_s\big(x^n f(x)\big) = (-1)^{m+1}\frac{d^n}{d\omega^n}[\mathcal{F}_c f(x)]$$

Where, $m = 1,2,3, \ldots$

Proof

Suppose $f(x)$ has a Fourier cosine and sine transforms:

$$\mathcal{F}_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos x\omega dx$$

And

$$\mathcal{F}_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin x\omega dx$$

Let, multiply $f(x)$ by x, then by definition; the Fourier sine transforms $xf(x)$ is,

$$\mathcal{F}_s(xf(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty xf(x)\sin x\omega dx$$

Let, $\mathcal{F}_s^* = \mathcal{F}_s(xf(x))$, then from the Fourier cosine transform we have:

$$\mathcal{F}_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos x\omega dx$$

by taking differentiation to both sides with respect to ω

$$\frac{d}{d\omega}[\mathcal{F}_c(f(x))] = \frac{d}{d\omega}[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos x\omega dx]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d}{d\omega}[f(x)\cos x\omega]dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\frac{d}{d\omega}[\cos x\omega]dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty -xf(x)\sin x\omega dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty xf(x)\sin x\omega dx$$

$$= -\mathcal{F}_s(xf(x))$$

$$= -\mathcal{F}_s^*$$

Therefore,

$$\mathcal{F}_s^*\big(f(x)\big) = -\frac{d}{d\omega}\big[\mathcal{F}_c\big(f(x)\big)\big]$$

Let, multiply $f(x)$ by x^2, then by definition; the Fourier cosine transforms of $x^2f(x))$ is,

$$\mathcal{F}_s(x^2f(x))) = \sqrt{\frac{2}{\pi}}\int_0^\infty x^2f(x))\sin x\omega dx$$

Then, from the Fourier sine transform we have:

$$\mathcal{F}_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin x\omega dx$$

by taking differentiation to both sides with respect to ω twice, we have:

$$\frac{d}{d\omega}\left[\mathcal{F}_s\big(f(x)\big)\right] = \frac{d}{d\omega}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin x\omega dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d}{d\omega}[f(x)\sin x\omega]dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\frac{d}{d\omega}[\sin x\omega]dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty xf(x)\cos x\omega dx$$

And

$$\frac{d^2}{d\omega^2}\left[\mathcal{F}_s\big(f(x)\big)\right] = \frac{d^2}{d\omega^2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin x\omega dx\right]$$

$$= \frac{d}{d\omega}\left[\frac{d}{d\omega}\left(\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin x\omega dx\right)\right]$$

$$= \frac{d}{d\omega}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty xf(x)\cos x\omega dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d}{d\omega}[xf(x)\cos x\omega]dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty -x^2f(x)\sin x\omega dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty x^2f(x)\sin x\omega dx$$

$$= -\mathcal{F}_s(x^2f(x))$$

$$= -\mathcal{F}_s^{**}$$

Therefore, $\mathcal{F}_s^{**} = -\frac{d^2}{d\omega^2}\left[\mathcal{F}_s\big(f(x)\big)\right]$

Continuing the process;

For n even and if $f(x)$ is multiplied by x^n, then the Fourier sine transform of $x^n f(x)$ is:

$$\mathcal{F}_s\big(x^n f(x)\big) = (-1)^{m+1} \frac{d^n}{d\omega^n}[\mathcal{F}_s f(x)]$$

Where, $m = 1,2,3, \ldots$

For n odd and if $f(x)$ is multiplied by x^n, then the Fourier sine transform of $x^n f(x)$ is:

$$\mathcal{F}_s\big(x^n f(x)\big) = (-1)^{m+1} \frac{d^n}{d\omega^n}[\mathcal{F}_c f(x)]$$

Where, $m = 1,2,3, \ldots$

3. Conclusions

The purpose of this paper is to provide a brief representation any function in integral form, Fourier cosine and sine transforms, after multiplying the given function by power functions; $x, x^2, x^3, \ldots x^n$ and provide the relation between Fourier Cosine transforms and Fourier Sine transforms. It is hoped that this implementation of the Fourier cosine and sine transforms will help representing the solutions of ODEs, PDEs, and integral equations that involving power functions terms in the integral form of simpler functions Cosine and Sine.

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