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A Study On The Simple Random Walk

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Abstract

An important class of Markov chain problems is the random walk problems. In a random walk the state of the Markov chain are the integers and the jumps of the chain from state i are only to neighbor states $i + 1$. There are many variations on this basic design. When the state space of the chain is finite it is sometimes called "the gambler's ruin problem". There are various martingale and Markov chain methods to analyze probabilistic characteristics of a simple random walk. In this study a simple random walk $S_n = X_1 + X_2 + \cdots + X_n$; $n \geq 1$; $S_0 = 0$ is defined and the first time (N) that this random walk visits the state 1 is analyzed by using generating functions. $E(N)$ is calculated in terms of the probabilities p and q.

Keywords: Markov chain; random walk; generating functions.

1. Introduction

Markov chain models are widely used in solving certain probabilistic problems and random walk problems are one of the important class of the Markov chain models. Excited random walks on integers is studied in [1]. A combination of artificial neural network and random walk models for financial time series forecasting is studied by [2]. Asymptotic Analysis of the Random-Walk Metropolis Algorithm on Ridged Densities is studied in [3]. In a study, prediction of exchange rates out of sample is investigated by some methods including random walk [4].

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In [5], diffusion and random walk processes are given in detail. There are various martingale and Markov chain methods to analyze probabilistic characteristics of a simple random walk. In this study a simple random walk $S_n = X_1 + X_2 + \cdots + X_n;\ n \geq 1;\ S_0 = 0$ and the first time (N) that this random walk visits the state 1 are defined. A recursion formula is obtained for the distribution of N and then this recursion formula is solved by using generating functions. After that $E(N)$ is calculated in terms of the probabilities p and q under some conditions, from the solution.

2. Simple Random Walk

Let $\{X_n, n \geq 1\}$ random variables be independent and identically distributed. If

i. $P(X_1 = 1) = p$ and $P(X_1 = -1) = 1 - p = q$; $0 \leq p, q \leq 1$; $p + q = 1$
ii. $X_0 = 0$
iii. $X_{n+1} - X_n$ is independent of $(X_0, X_1, \dots, X_n)$ for all $n \in \mathbb{N}$

Then the simple random walk process $\{S_n, n \geq 1\}$ is defined as,

$$S_n = X_1 + X_2 + \cdots + X_n;\ n \geq 1;\ S_0 = 0$$

It is called "simple" from the fact that the size of each step is fixed (equal to 1) and it is only to one direction chosen randomly. If $p = q = 1/2$ then the random walk is symmetric. The random walk $\{S_n\}$ is mostly used in modelling a gambling problem: in a game a gambler wins a dollar with probability p or loses a dollar with probability $q = 1 - p$. Although there are a number of Markov chain and martingale methods to analyze $\{S_n\}$, in this study generating functions are used to obtain some characteristics of $\{S_n\}$.

3. Obtaining probabilistic characteristics of the Simple Random Walk

Let $N = inf\{n \geq 1: S_n = 1\}$ be the first time that the random walk jumps to the state 1, in other words that the gambler is ahead for the first time. The aim is to obtain the distribution of N. To do this, first a difference equation is constructed and then this equation is solved using generating functions.

Let,

$$\xi_n = P(N = n), n \geq 0$$

then, $\xi_0 = 0$ and $\xi_1 = p$. If $n \geq 2$, then in order to for the random walk moves from 0 to 1 in n steps, the very first step must be to -1, which has the probability q. After that, the walk must be back to 0, assume that this takes j steps. Then, the probability that the random walk moves from -1 to 0 in j step is ξ_j. After reaching the state 0, the random walk must go to state 1, lets say this takes k steps, hence this probability is ξ_k and $1 + j + k = n$. Thus, the following recursion formula is obtained [6],

$$\xi_0 = 0\ , \xi_1 = p$$

$$\xi_n = \sum_{j=1}^{n-2} q\xi_j\xi_{n-j-1}\,,\ \ n \geq 2. \qquad (1)$$

In order to solve equation (1), both sides of equality is multiplied by s^n and sum over n . Let

$$\Phi(s) = \sum_{n=0}^{\infty} \xi_n s^n,$$

then we have,

$$\sum_{n=2}^{\infty} \xi_n s^n = \sum_{n=2}^{\infty} \left(\sum_{j=1}^{n-2} q\xi_j\xi_{n-j-1} \right) s^n$$

$$= \sum_{n=2}^{\infty} \left(\sum_{j=0}^{n-2} q\xi_j\xi_{n-j-1} \right) s^n. \qquad (2)$$

Reversing the order of summation and setting $m = n - j - 1$ yields:

$$= \sum_{j=0}^{\infty} \left(\sum_{n=j+2}^{\infty} \xi_{n-j-1} s^{n-j-1} \right) \xi_j s^j qs$$

$$= \sum_{j=0}^{\infty} \left(\sum_{m=1}^{\infty} \xi_m s^m \right) \xi_j s^j qs$$

$$= \sum_{j=0}^{\infty} \Phi(s)\xi_j s^j qs = qs\Phi(s) \sum_{j=0}^{\infty} \xi_j s^j$$

$$= qs\Phi^2(s).$$

The left hand side of equation (2) is:

$$\sum_{n=1}^{\infty} \xi_n s^n - \xi_1 s = \Phi(s) - ps,$$

thus,

$$\Phi(s) - ps = qs\Phi^2(s). \qquad (3)$$

The equation (3) is a quadratic equation for the unknown $\Phi(s)$, and solution of this equation is,

$$\Phi(s) = \frac{\left(1 \pm \sqrt{1-4pqs^2}\right)}{2qs} \tag{4}$$

The solution with " $+$ " sign is probabilistically not acceptable, thus:

$$\Phi(s) = \frac{1-\sqrt{1-4pqs^2}}{2qs}, \qquad 0 \le s \le 1 \tag{5}$$

Equation (5) can be expanded in order to have an explicit solution for $\{\xi_n\}$ by the Binomial Theorem [6]. From the definition of $\Phi(s)$ we know that,

$$\Phi(s) = \sum_{n=1}^{\infty} \xi_n s^n$$

and by expanding equation (5), we have

$$\Phi(s) = \left(1 - \sum_{j=0}^{\infty} \binom{\frac{1}{2}}{j} (-1)^j (4pqs^2)^j \right) \Big/ 2qs$$

$$= \sum_{j=1}^{\infty} \binom{\frac{1}{2}}{j} (-1)^{j+1} (4pq)^j s^{2j} \Big/ 2qs$$

$$= \sum_{j=1}^{\infty} \binom{\frac{1}{2}}{j} (-1)^{j+1} \frac{(4pq)^j}{2q} s^{2j-1}$$

$$= (.)s + (.)s^3 + (.)s^5 + \cdots$$

thus, for the odd indices we have for

$$\xi_{2j-1} = \binom{\frac{1}{2}}{j} (-1)^{j+1} (4pq)^j \Big/ 2q\,, \quad j \ge 1,$$

and

$$\xi_{2j} = 0$$

for the even indices. Now using equation (5) we compute,

$$P(N < \infty) = \Phi(1) = \left(1 - \sqrt{1-4p(1-p)}\right)\Big/ 2q$$

$$= (1 - |p - q|)/2q$$

$$P(N < \infty) = \begin{cases} 1 & , \quad p \geq q \\ p/q & , \quad p < q \end{cases}$$

If $p < q$ then,

$$P(N = \infty) = 1 - (p/q) > 0$$

When $P(N = \infty) > 0$, by the definition we get $E(N) = \infty$. On the other hand, when $p \geq q$ $E(N) = \Phi'(1)$ is computed by differentiating equation (5):

$$\Phi'(s) = \frac{2qs\left(-\frac{1}{2}(1 - 4pqs^2)^{-1/2}\right)(-8pqs) - \left(1 - \sqrt{1 - 4pqs^2}\right)2q}{4q^2s^2}$$

so that,

$$E(N) = \Phi'(1) = \frac{2p}{|p - q|} - \frac{(1 - |p - q|)}{2q}$$

and finally,

$$E(N) = \begin{cases} \infty & , \quad p = q = 1/2 \\ (p - q)^{-1} & , \quad p > q \end{cases}$$

4. Conclusion and Discussion

There are various martingale and Markov chain methods to analyze probabilistic characteristics of a simple random walk. In this study a simple random walk $S_n = X_1 + X_2 + \cdots + X_n$; $n \geq 1$; $S_0 = 0$ and the first time (N) that this random walk visits the state 1 are defined. A recursion formula is obtained for the distribution of N and then this recursion formula is solved by using generating functions. After that $E(N)$ is calculated in terms of the probabilities p and q under some conditions, from the solution. For further studies, single step transition probability matrix can be used to obtain a stationary distribution of the random walk.

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