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Positive $P_{0,1}$ Matrix Completion Problem for Order Four Digraphs with Zero, One, Two and Three Arcs

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Abstract

In this paper the positive $P_{0,1}$ matrix completion problem for digraphs of order 4 with zero, one, two and three arcs isconsidered. It is shown that a partial positive $P_{0,1}$ matrix specifying digraphs of order 4 with zero, one, two and three arcs have completion.

Keywords: Partial matrix; principal sub matrix; principal minor; matrix completion; positive $P_{0,1}$ matrix; symmetric pattern; isomorphism.

1. Introduction

A real $\boldsymbol{n} \times \boldsymbol{n}$ matrix is called a positive $\boldsymbol{P_{0,1}}$ matrix if all its entries are positive and all its principal minors are non negative. A partial positive $\boldsymbol{P_{0,1}}$ matrix is a partial matrix in which all fully specified principal sub matrices are positive $\boldsymbol{P_{0,1}}$ matrices [2].

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A pattern for $\boldsymbol{n} \times \boldsymbol{n}$ matrices is a list of positions of an $\boldsymbol{n} \times \boldsymbol{n}$ matrix and it is a subset of $(\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}) \times (\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n})$. A positionally symmetric pattern is a pattern in which $(\boldsymbol{i}, \boldsymbol{j})$ is in the pattern if and only if $(\boldsymbol{j}, \boldsymbol{i})$ is in the pattern. A partial matrix specifies a pattern if its entries lie exactly in those positions listed in the pattern [1].

A graph $G = (V, E)$ is a finite non empty set made up of vertices and edges. A sub graph of a graph $G = (V, E)$ is a graph $H = (V', E')$ where V' is a subset of V and E' is a subset of E [3].A digraph $D = (V, A)$ is a finite set of positive integers V whose members are called vertices and a set E of ordered pairs (u, v) of vertices called arcs (directed edges). A digraph which contains all possible arcs between its vertices is called a complete digraph. A sub digraph of a digraph $D = (V, A)$ is a digraph $H = (V', A')$ where V' is a subset of V and A' is a subset of A [3].

Graph theory is considered to play an important role in the study of matrix completion problems. A positionally symmetric pattern for $n \times n$ matrices that include all diagonal positions can be represented by means of a graph $G = (V, E)$ on n vertices, where $V = \{1,2, \ldots, n\}$ and E is the edge set. For $1 \leq i, j \leq n$, the edge (i, j) belongs to E if and only if the ordered pair (j, i) is also in the pattern. A non symmetric pattern for $n \times n$ matrices that includes all diagonal positions, is best described by means of a digraph $D = (V, E)$ on n vertices. Therefore the directed arc (i, j) , $1 \leq i, j \leq n$, is in the arc set E if and only if the ordered pair (i, j) is in the pattern [1].

In many situations it is convenient to permute entries of a partial matrix. A permutation matrix P is obtained by interchanging rows of an identity matrix, therefore all its entries are either 1 or 0 and there is exactly one 1 in each row and each column of the matrix. A permutation similarity of A is a product P^T , where P is a permutation matrix. This is represented on a digraph by renumbering the vertices. As a result of the following lemma, we can permute a partial positive $P_{0,1}$ matrix and hence renumber vertices as is convenient.

Lemma 1.1: The class of P_0 matrices is closed under permutation [1].

Remark: Two digraphs D_1 and D_2 are isomorphic if it is possible to obtain D_2 from D_1 by renumbering the vertices[1].

The matrix completion problem deals with determining whether or not a completion of a partial matrix exists for a certain class of matrices. A description of circumstances is sought in which choices for the unspecified entries may be made so that the resulting matrix is of the desired class.

In recent years, graphs and digraphs have been used very effectively to study matrix completion problems. Classes of symmetric matrices like P matrices have been studied by means of their graphs while matrices without positional symmetry have been studied by means of their digraphs.

Since the class of matrices we are studying is not symmetric, we use digraphs. A partial matrix specifies a digraph if its specified entries are exactly the arcs on the digraph. The digraphs used in this study include all diagonal positions [3].To obtain a partial matrix, an arc on the digraph represents a specified entry on the partial matrix, denoted by a_{ij} , while a missing arc on the digraph represents an unspecified entry on the partial matrix.

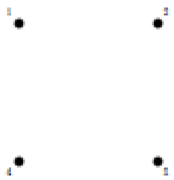
Some study has been done on positive $P_{0,1}$matrix completion. In [2], Hogben established that all digraphs of order 2 have positive $P_{0,1}$ completion. In [4], J. Mutembei and his colleagues showed that all digraphs of order 3 with zero, one ,two and three arcs have positive $P_{0,1}$ completion. In [5], J..Mutembei and his colleagues also showed that a digraph of order 3 which omits only one arc has no completion. However, much of the study on this class of matrices has not been done to solve the class of matrices.

In this paper, the positive $P_{0,1}$ matrix completion problem is discussed for order 4 digraphs with zero, one, two and three arcs. Throughout this paper, the entries of a partial matrix A are denoted as follows; d_i denotes a specified diagonal entry, a_{ij} denotes a specified non diagonal entry, and x_{ij} an unspecified entry, $1 \leq i,j \leq n$.

All order 4 digraphs with zero, one, two and three arcs are considered and 4×4 partial matrices specifying these digraphs constructed. The construction of the digraphs is guided by the graphs of order 4 as given in [6].

CASE 1: p = 4, q = 0

There is only one digraph in this category up to isomorphism.



Let $A = \begin{pmatrix} d_1 & x_{12} & x_{13} & x_{14} \\ x_{21} & d_2 & x_{23} & x_{24} \\ x_{31} & x_{32} & d_3 & x_{34} \\ x_{41} & x_{42} & x_{43} & d_4 \end{pmatrix}$ be the partial positive $P_{0,\ 1}$ - matrix representing the digraph above. By definition of the completion $d_1 > 0$ $d_2 > 0, d_3 > 0$ and $d_4 > 0$

Considering the principal minors of A, we have; Completion of the 2×2 and 3×3 submatrices follow from [4], therefore we show completion of the 4×4 matrix A.

$$|A| = d_1[d_2d_3d_4 + x_{23}x_{34}x_{42} + x_{24}x_{43}x_{32} - x_{24}d_3x_{42} - x_{34}x_{43}d_2 - d_4x_{32}x_{23}]$$

$$-x_{12}[x_{21}d_3d_4 + x_{23}x_{34}x_{41} + x_{24}x_{42}x_{31} - x_{24}d_3x_{41} - x_{34}x_{43}x_{21} - d_4x_{31}x_{23}]$$

$$+x_{13}[x_{21}x_{32}d_4 + d_2x_{34}x_{41} + x_{24}x_{42}x_{31} - x_{24}x_{32}x_{41} - x_{34}x_{42}x_{21} - d_4x_{31}d_2]$$

$$-x_{14}[x_{21}x_{32}x_{43} + d_2d_3x_{41} + x_{23}x_{42}x_{31} - x_{23}x_{32}x_{41} - d_2x_{31}x_{43} - x_{21}x_{42}d_3]$$

The first term $d_1 d_2 d_3 d_4$ is positive, therefore setting the specified entries to be sufficiently large and unspecified entries to be sufficiently small gives $|A| \geq 0$, hence the matrix A has completion.

Table 1

Sub matrix	**Determinant**
$A(1,2)$	$d_1\ d_2 - x_{12}x_{21}$
$A(1,3)$	$d_1 d_3 - x_{13}x_{31}$
$A(1,4)$	$d_1 d_4 - x_{14}x_{41}$
$A(2,3)$	$d_2 d_3 - x_{23}x_{32}$
$A(2,4)$	$d_2 d_4 - x_{24}x_{42}$
$A(3,4)$	$d_3 d_4 - x_{34}x_{43}$
$A(1,2,3)$	$d_1 d_2 d_3 + x_{12}x_{23}x_{31} + x_{13}x_{32}x_{21} - x_{13}x_{31}d_2 - x_{23}x_{32}d_1 - d_3 x_{21}x_{12}$
$A(1,2,4)$	$d_1 d_2 d_4 + x_{12}x_{24}x_{41} + x_{14}x_{42}x_{21} - x_{14}d_2 x_{41} - x_{24}x_{42}d_1 - d_4 x_{21}x_{12}$
$A(1,3,4)$	$d_1 d_3 d_4 + x_{13}x_{34}x_{41} + x_{14}x_{43}x_{31} - x_{14}d_3 x_{41} - x_{34}x_{43}d_1 - d_4 x_{31}x_{13}$
$A(2,3,4)$	$d_2 d_3 d_4 + x_{23}x_{34}x_{42} + x_{24}x_{43}x_{32} - x_{24}x_{42}d_3 - x_{34}x_{43}d_2 - d_4 x_{32}x_{24}$
$A(1,2,3,4)$	$d_1[d_2 d_3 d_4 + x_{23}x_{34}x_{42} + x_{24}x_{43}x_{32} - x_{24}d_3 x_{42} - x_{34}x_{43}d_2 - d_4 x_{32}x_{23}]$ $-x_{12}[x_{21}d_3 d_4 + x_{23}x_{34}x_{41} + x_{24}x_{42}x_{31} - x_{24}d_3 x_{41} - x_{34}x_{43}x_{21} - d_4 x_{31}x_{23}]$ $+x_{13}[x_{21}x_{32}d_4 + d_2 x_{34}x_{41} + x_{24}x_{42}x_{31} - x_{24}x_{32}x_{41} - x_{34}x_{42}x_{21} - d_4 x_{31}d_2]$ $-x_{14}[x_{21}x_{32}x_{43} + d_2 d_3 x_{41} + x_{23}x_{42}x_{31} - x_{23}x_{32}x_{41} - d_2 x_{31}x_{43} - x_{21}x_{42}d_3]$

CASE 2: p = 4, q = 1 : There is only one digraph in this category upto isomorphism.

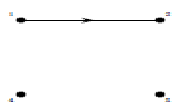


Figure 1

$$\text{Let } A = \begin{pmatrix} d_1 & a_{12} & x_{13} & x_{14} \\ x_{21} & d_2 & x_{23} & x_{24} \\ x_{31} & x_{32} & d_3 & x_{34} \\ x_{41} & x_{42} & x_{43} & d_4 \end{pmatrix}$$

be the partial positive $P_{0,1}$ - matrix representing the digraph above. By definition of the completion $d_1 > 0\ d_2 > 0, d_3 > 0, d_4 > 0$ and $a_{12} > 0$

Considering the principal minors of A, we have: Completion of the 2×2 and 3×3 submatrices follow from [4], therefore we show completion of the 4×4matrix$A$.

Table 2

Sub matrix	Determinant
$A(1,2)$	$d_1\ d_2 - a_{12}x_{21}$
$A(1,3)$	$d_1d_3 - x_{13}x_{31}$
$A(1,4)$	$d_1d_4 - x_{14}x_{41}$
$A(2,3)$	$d_2d_3 - x_{23}x_{32}$
$A(2,4)$	$d_2d_4 - x_{24}x_{42}$
$A(3,4)$	$d_3d_4 - x_{34}x_{43}$
$A(1,2,3)$	$d_1d_2d_3 + a_{12}x_{23}x_{31} + x_{13}x_{32}x_{21} - x_{13}x_{31}d_2 - x_{23}x_{32}d_1 - d_3x_{21}x_{12}$
$A(1,2,4)$	$d_1d_2d_4 + a_{12}x_{24}x_{41} + x_{14}x_{42}x_{21} - x_{14}d_2x_{41} - x_{24}x_{42}d_1 - d_4x_{21}a_{12}$
$A(1,3,4)$	$d_1d_3d_4 + x_{13}x_{34}x_{41} + x_{14}x_{43}x_{31} - x_{14}d_3x_{41} - x_{34}x_{43}d_1 - d_4x_{31}x_{13}$
$A(2,3,4)$	$d_2d_3d_4 + x_{23}x_{34}x_{42} + x_{24}x_{43}x_{32} - x_{24}x_{42}d_3 - x_{34}x_{43}d_2 - d_4x_{32}x_{24}$
$A(1,2,3,4)$	$d_1[d_2d_3d_4 + x_{23}x_{34}x_{42} + x_{24}x_{43}x_{32} - x_{24}d_3x_{42} - x_{34}x_{43}d_2 - d_4x_{32}x_{23}]$ $-a_{12}[x_{21}d_3d_4 + x_{23}x_{34}x_{41} + x_{24}x_{42}x_{31} - x_{24}d_3x_{41} - x_{34}x_{43}x_{21} - d_4x_{31}x_{23}]$ $+x_{13}[x_{21}x_{32}d_4 + d_2x_{34}x_{41} + x_{24}x_{42}x_{31} - x_{24}x_{32}x_{41} - x_{34}x_{42}x_{21} - d_4x_{31}d_2]$ $-x_{14}[x_{21}x_{32}x_{43} + d_2d_3x_{41} + x_{23}x_{42}x_{31} - x_{23}x_{32}x_{41} - d_2x_{31}x_{43} - x_{21}x_{42}d_3]$

$$|A| = d_1[d_2d_3d_4 + x_{23}x_{34}x_{42} + x_{24}x_{43}x_{32} - x_{24}d_3x_{42} - x_{34}x_{43}d_2 - d_4x_{32}x_{23}]$$

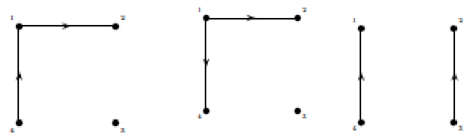
$$-a_{12}[x_{21}d_3d_4 + x_{23}x_{34}x_{41} + x_{24}x_{42}x_{31} - x_{24}d_3x_{41} - x_{34}x_{43}x_{21} - d_4x_{31}x_{23}]$$

$$+x_{13}[x_{21}x_{32}d_4 + d_2x_{34}x_{41} + x_{24}x_{42}x_{31} - x_{24}x_{32}x_{41} - x_{34}x_{42}x_{21} - d_4x_{31}d_2]$$

$$-x_{14}[x_{21}x_{32}x_{43} + d_2d_3x_{41} + x_{23}x_{42}x_{31} - x_{23}x_{32}x_{41} - d_2x_{31}x_{43} - x_{21}x_{42}d_3]$$

Since the term $d_1d_2d_3d_4$ is positive, therefore setting the specified entries to be sufficiently large and unspecified entries to be sufficiently small gives $|A| \geq 0$ hence the matrix A has completion.

CASE 3: p = 4, q = 2: The digraphs in this category upto isomorphism are;



(i) (ii) (iii)

Figure 2

Sub case: 3(i) :

$$\text{Let } A = \begin{pmatrix} d_1 & a_{12} & x_{13} & x_{14} \\ x_{21} & d_2 & x_{23} & x_{24} \\ x_{31} & x_{32} & d_3 & x_{34} \\ a_{41} & x_{42} & x_{43} & d_4 \end{pmatrix}$$

be the partial positive $P_{0,1}$ - matrix representing the digraph 3(i) above. By definition of the completion $d_1 > 0$, $d_2 > 0, d_3 > 0, d_4 > 0$, $a_{12} > 0$ and $a_{41} > 0$.

Considering the principal minors of A, we have:

Table 3

Sub matrix	**Determinant**
$A(1,2)$	$d_1\ d_2 - a_{12}x_{21}$
$A(1,3)$	$d_1 d_3 - x_{13}x_{31}$
$A(1,4)$	$d_1 d_4 - a_{14}x_{41}$
$A(2,3)$	$d_2 d_3 - x_{23}x_{32}$
$A(2,4)$	$d_2 d_4 - x_{24}x_{42}$
$A(3,4)$	$d_3 d_4 - x_{34}x_{43}$
$A(1,2,3)$	$d_1d_2d_3 + a_{12}x_{23}x_{31} + x_{13}x_{32}x_{21} - x_{13}x_{31}d_2 - x_{23}x_{32}d_1 - d_3x_{21}x_{12}$
$A(1,2,4)$	$d_1d_2d_4 + a_{12}x_{24}x_{41} + a_{14}x_{42}x_{21} - a_{14}d_2x_{41} - x_{24}x_{42}d_1 - d_4x_{21}a_{12}$
$A(1,3,4)$	$d_1d_3d_4 + x_{13}x_{34}x_{41} + a_{14}x_{43}x_{31} - a_{14}d_3x_{41} - x_{34}x_{43}d_1 - d_4x_{31}x_{13}$
$A(2,3,4)$	$d_2d_3d_4 + x_{23}x_{34}x_{42} + x_{24}x_{43}x_{32} - x_{24}x_{42}d_3 - x_{34}x_{43}d_2 - d_4x_{32}x_{24}$
$A(1,2,3,4)$	$d_1[d_2d_3d_4 + x_{23}x_{34}x_{42} + x_{24}x_{43}x_{32} - x_{24}d_3x_{42} - x_{34}x_{43}d_2 - d_4x_{32}x_{23}]$ $-a_{12}[x_{21}d_3d_4 + x_{23}x_{34}x_{41} + x_{24}x_{42}x_{31} - x_{24}d_3x_{41} - x_{34}x_{43}x_{21} - d_4x_{31}x_{23}]$ $+x_{13}[x_{21}x_{32}d_4 + d_2x_{34}x_{41} + x_{24}x_{42}x_{31} - x_{24}x_{32}x_{41} - x_{34}x_{42}x_{21} - d_4x_{31}d_2]$ $-a_{14}[x_{21}x_{32}x_{43} + d_2d_3x_{41} + x_{23}x_{42}x_{31} - x_{23}x_{32}x_{41} - d_2x_{31}x_{43} - x_{21}x_{42}d_3]$

Completion of the 2×2 and 3×3 submatrices follow from [4], therefore we show completion of the 4×4 matrix A.

$$|A| = d_1[d_2d_3d_4 + x_{23}x_{34}x_{42} + x_{24}x_{43}x_{32} - x_{24}d_3x_{42} - x_{34}x_{43}d_2 - d_4x_{32}x_{23}]$$

$$-a_{12}[x_{21}d_3d_4 + x_{23}x_{34}x_{41} + x_{24}x_{42}x_{31} - x_{24}d_3x_{41} - x_{34}x_{43}x_{21} - d_4x_{31}x_{23}]$$

$+x_{13}[x_{21}x_{32}d_4 + d_2x_{34}x_{41} + x_{24}x_{42}x_{31} - x_{24}x_{32}x_{41} - x_{34}x_{42}x_{21} - d_4x_{31}d_2]$

$-a_{14}[x_{21}x_{32}x_{43} + d_2d_3x_{41} + x_{23}x_{42}x_{31} - x_{23}x_{32}x_{41} - d_2x_{31}x_{43} - x_{21}x_{42}d_3]$

CASE 4: p=4, q=3: The only digraph in this category upto isomorphism are;

Since the term $d_1d_2d_3d_4$ is positive, therefore setting the specified entries to be sufficiently large and unspecified entries to be sufficiently small gives $|A| \geq 0$, hence the matrix A has completion. By similar argument, partial matrices specifying the digraphs 3(ii) and 3(iii) can be shown to have completion. Therefore all digraphs of order 4 with two arcs have positive $P_{0,1}$ completion.

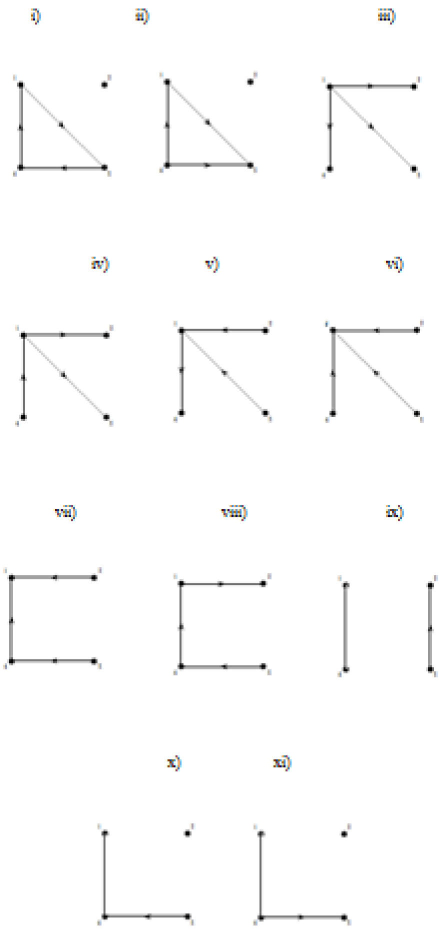


Figure 3

Sub case: 4(i)

Let $A = \begin{pmatrix} d_1 & x_{12} & a_{13} & x_{14} \\ x_{21} & d_2 & x_{23} & x_{24} \\ x_{31} & x_{32} & d_3 & a_{34} \\ a_{41} & x_{42} & x_{43} & d_4 \end{pmatrix}$

be the partial positive $P_{0,1}$ - matrix representing the digraph 4(i) above. By definition of the completion $d_1 > 0$, $d_2 > 0, d_3 > 0, d_4 > 0$, $a_{13} > 0, a_{34} > 0$ and $a_{41} > 0$.

Considering the principal minors of A, we have:

Table 4

Sub matrix	**Determinant**
$A(1,2)$	$d_1\ d_2 - x_{12}x_{21}$
$A(1,3)$	$d_1d_3 - a_{13}x_{31}$
$A(1,4)$	$d_1d_4 - x_{14}a_{41}$
$A(2,3)$	$d_2d_3 - x_{23}x_{32}$
$A(2,4)$	$d_2d_4 - x_{24}x_{42}$
$A(3,4)$	$d_3d_4 - a_{34}x_{43}$
$A(1,2,3)$	$d_1d_2d_3 + x_{12}x_{23}x_{31} + a_{13}x_{32}x_{21} - a_{13}x_{31}d_2 - x_{23}x_{32}d_1 - d_3x_{21}x_{12}$
$A(1,2,4)$	$d_1d_2d_4 + x_{12}x_{24}a_{41} + x_{14}x_{42}x_{21} - x_{14}d_2a_{41} - x_{24}x_{42}d_1 - d_4x_{21}x_{12}$
$A(1,3,4)$	$d_1d_3d_4 + a_{13}a_{34}a_{41} + x_{14}x_{43}x_{31} - x_{14}d_3a_{41} - a_{34}x_{43}d_1 - d_4x_{31}a_{13}$
$A(2,3,4)$	$d_2d_3d_4 + x_{23}a_{34}x_{42} + x_{24}x_{43}x_{32} - x_{24}x_{42}d_3 - a_{34}x_{43}d_2 - d_4x_{32}x_{24}$
$A(1,2,3,4)$	$d_1[d_2d_3d_4 + x_{23}a_{34}x_{42} + x_{24}x_{43}x_{32} - x_{24}d_3x_{42} - a_{34}x_{43}d_2 - d_4x_{32}x_{23}]$ $-x_{12}[x_{21}d_3d_4 + x_{23}a_{34}a_{41} + x_{24}x_{42}x_{31} - x_{24}d_3x_{41} - a_{34}x_{43}x_{21} - d_4x_{31}x_{23}]$ $+a_{13}[x_{21}x_{32}d_4 + d_2a_{34}a_{41} + x_{24}x_{42}x_{31} - x_{24}x_{32}a_{41} - a_{34}x_{42}x_{21} - d_4x_{31}d_2]$ $-x_{14}[x_{21}x_{32}x_{43} + d_2d_3a_{41} + x_{23}x_{42}x_{31} - x_{23}x_{32}a_{41} - d_2x_{31}x_{43} - x_{21}x_{42}d_3]$

Completion of the 2×2 and 3×3 submatrices follow from [4], therefore we show completion of the 4×4 matrix A.

$$|A| = d_1[d_2d_3d_4 + x_{23}a_{34}x_{42} + x_{24}x_{43}x_{32} - x_{24}d_3x_{42} - a_{34}x_{43}d_2 - d_4x_{32}x_{23}]$$

$$-x_{12}[x_{21}d_3d_4 + x_{23}a_{34}a_{41} + x_{24}x_{42}x_{31} - x_{24}d_3x_{41} - a_{34}x_{43}x_{21} - d_4x_{31}x_{23}]$$

$$+a_{13}[x_{21}x_{32}d_4 + d_2a_{34}a_{41} + x_{24}x_{42}x_{31} - x_{24}x_{32}a_{41} - a_{34}x_{42}x_{21} - d_4x_{31}d_2]$$

$$-x_{14}[x_{21}x_{32}x_{43} + d_2d_3a_{41} + x_{23}x_{42}x_{31} - x_{23}x_{32}a_{41} - d_2x_{31}x_{43} - x_{21}x_{42}d_3]$$

Since the term $d_1d_2d_3d_4$ is positive, it is possible to set the specified entries to be sufficiently large and unspecified entries to be sufficiently small to give $|A| \geq 0$, hence the matrix A has completion.

By similar argument, partial matrices specifying all the digraphs above can be shown to have completion. Therefore all digraphs of order 4 with three arcs have positive $P_{0,1}$ completion.

4. Conclusion and Recommendation.

It has been shown that all order 4 digraphs which are null or have one, two or three arcs have positive $P_{0,1}$ completion. This agrees with the results of obtained from order 3 digraphs [4]. This implies that digraph characteristics that lead to completion of partial matrices specifying order 3 digraphs are inherited by order 4 digraphs. It would be interesting to extend this research to determine whether this property can be generalized upto $n \in \mathbb{Z}^{+}$.

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