International Journal of Sciences: Basic and Applied Research (IJSBAR)

ISSN 2307-4531
(Print & Online)

http://gssrr.org/index.php?journal=JournalOfBasicAndApplied

Optimization of Heterogeneous Parallel Queue Systems with Heavy Traffic

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Abstract

This paper investigates heterogeneous and homogeneous parallel queue systems with heavy traffic. The steady-state probabilities are obtained in $M/M/n$ queue systems with heavy traffic when the number of customers in the system is larger than the number of parallel service channels of the system. The steady-state probabilities are ignored when the number of customers in system at any given time is smaller than the number of service channels since these probabilities are very small. Similarly, heavy working $M/\vec{M}/n$ queue system is analyzed ignoring the steady-state probabilities when the number of customers in system is smaller than the number of service channels. In heavy working $M/M/n$ and $M/\vec{M}/n$ queue systems it is shown that both systems have same performance measures and none of these two queue systems have advantages against each other under condition $\mu_1 + \mu_2 + \cdots + \mu_n = c$.

Keywords: Queue systems; Steady-state distrubition; Poisson arrivals; Markov chains; Homogeneous service channel; Heterogeneous service channel.

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1. Introduction

Waiting for a service such as at restaurants, hospitals, shopping malls, traffic or in storage processes, is a part of daily life [4]. There are many studies in literature considering queue systems with parallel service channels. [8] Investigated blocked Markov queue system with two heterogeneous service channels. In [5] the author analysed a two channel queue system with waiting line in which the arrivals are Poisson distributed and service times of the channels are different. On the other hand departures in $M/M/n$ queue system are studied in [1]. Queueing system with heterogeneous service channels firstly studied by [2], in this study the steady-state probabilities of the system are determined. Reference [9] showed that $M/M/1$ queue system with service parameter $n\mu$ works better than $M/M/n$ system with service parameter μ. The $M/G/n/0$ queueing system with loss probability and general service distribution is analysed by [6], in this study the loss probability of system is minimized. In another study, Reference [7] compared the homogeneous $M/M/n$ systems with heterogeneous $M/\vec{M}/n$ systems and showed that homogeneous system works faster than heterogeneous system when the total service capacity is fixed. In this paper $M/\vec{M}/n$ queueing system is analysed and performance measures of the system are obtained when the traffic is heavy (number of arrivals are greater than the number of departures). It is shown that performance measures of a heavy traffic $M/\vec{M}/n$ queue system are constant when the total service capacity is fixed. Also the results of analysed heavy traffic queue system are evaluated.

2. The Analyze of M/M ⃗/n Queue System

2.1. Definition of the System

$M/\vec{M}/n$ queue system is consisted of n service channels numbered as $1,2,\dots,n$. The arrivals are according to Poisson distribution with parameter λ and the service time of k^{th} service channel is exponentially distributed with parameter μ_k. Since the mean service time of each service channel is different, this queue system is heterogeneous. The inter-arrival time and service time are independently distributed random variables. A new arriving customer chooses one of the available service channel with equal probabilities. If all service channels are busy the customer waits in queue according to the FIFO (first in first out) discipline. The number of customers in queue can be countable-infinite and the source of arrivals is infinite.

2.2. Steady-state probabilities of $M/\vec{M}/n$ queue system

The steady-state probabilites of $M/\vec{M}/n$ system is given as follows:

$$p_k = \begin{cases} p_0 \dfrac{(n-k)!}{n!} \sum \rho_{i_1} \cdot \rho_{i_2} \dots \rho_{i_k} & ; k < n \\ p_0 \dfrac{\rho_1 \cdot \rho_2 \dots \rho_n}{n!} \left(\dfrac{\lambda}{c}\right)^{k-n} & ; k \geq n \end{cases} \tag{1}$$

where μ_i is the service time of the i^{th} service channel, λ is the arrival rate and $\rho_i = \lambda/\mu_i$ is the traffic density of the queue system (Shahbazov, 1996).

In equation (1), $E_k = E_k\ (\rho_1, \rho_2, \dots, \rho_n\ )$ is a k −symmetric function and defind as,

$$E_k = \sum_{Q_{k,n}} \rho_{i_1} \cdot \rho_{i_2} \dots \rho_{i_k}$$

in which $Q_{k,n}$ is a set of k variables out of n service channels $i_1, i_2, \dots, i_k;\ 1 \le i_1 < \cdots < i_k \le n$. The p_k probabilities in equation (1) are written in terms of E_k as following:

$$p_k = \begin{cases} p_0 \dfrac{(n-k)!}{n!} E_k & ; k < n \\ \dfrac{E_n}{n!} \rho^{k-n} p_0 = \ p_n \rho^{k-n} & ; k \ge n \end{cases} \tag{2}$$

where,

$$\frac{1}{p_0} = \frac{1}{n!}\left[\sum_{k=0}^{n-1} (n-k)!\, E_k + \frac{E_n}{1-\rho}\right] \tag{3}$$

[7].

Let p_w be the probability that the service channels are busy. Then,

$$\frac{1}{p_w} = 1 + (1-\alpha) \sum_{k=1}^{n} k!\, e_k \tag{4}$$

under condition $\mu_1 + \mu_2 + \cdots + \mu_n > \lambda$. Also the inequality,

$$\rho^n \le p_w \le \rho \tag{5}$$

where

$$\rho = \frac{\lambda}{\mu_1 + \mu_2 + \cdots + \mu_n} = \frac{\lambda}{c}$$

holds [7]

In equation (4),

$$e_k = e_k \left({}^{\mu_1}/_{\lambda}, {}^{\mu_2}/_{\lambda}, \dots, {}^{\mu_n}/_{\lambda}\right)$$

is a k −symmetric function (Mitrinovic, 1970).

2.3. Finding the parameters of the system $M/\overrightarrow{M}/n$

Let $E(\xi_q), E(w)$ and $E(w_q)$ be the mean number of customers in queue, mean waiting time in queue and mean waiting number in the system, respectively. Using p_k probabilities we obtain these performance measures $\left(E(\xi_q),\ E(w),\ E(w_q)\right)$ in terms of p_w as following:

$$E(\xi_q) = p_w \cdot \frac{\rho}{1-\rho} \tag{7}$$

$$E(w_q) = \frac{1}{\lambda}E(\xi_q) = \frac{1}{\lambda}\cdot\frac{\lambda}{c-\lambda}\cdot p_w = \frac{p_w}{(c-\lambda)} \tag{8}$$

$$E(w) = E(w_q) + E(\eta) = \frac{p_w}{(c-\lambda)} + \frac{n}{c} \tag{9}$$

where η is the service time.

3. Parallel $M/\vec{M}/n$ Queue System with Heavy Traffic

In queue systems with heavy traffic, the number of customers (k) is greater than the number of service channels (n). In other words there always exist a queue in the system, hence a new incoming customer has to wait for service. Considering the steady-state probabilities of this queue system, the probability that the number of customers (k) is smaller than the number of service channels (n) is ignored.

3.1. Definition of heavy traffic $M/\vec{M}/n$ queueing system

In this queueing system the inter- arrival times exponentially distributed with parameter λ and the number of service channles in the system is $n; n \geq 1$. The service time of these channels are exponentially distributed with means $\mu_k^{-1}\ \ k = 1,2,\dots,n$ respectively.

3.2. Finding steady-state probabilities of $M/\vec{M}/n$ queueing system

In $M/\vec{M}/n$ queueing system with heavy traffic the probability that the number of customers is smaller than the number of service channels is so small, hence the system analyzed considering this probability is zero. If the traffic density of the system tends to 1 then the probability that the queueing system is empty tends to zero. Hence the probability that the number of customers is smaller than the number of service channels tends to zero.

$$\lim_{\rho\to 1}(p_0) = \lim_{\rho\to 1}\left\{\frac{1}{\frac{1}{n!}\left[\sum_{k=0}^{n-1}(n-k)!\,E_k + \frac{E_n}{1-\frac{\lambda}{c}}\right]}\right\} = 0$$

$$p_k \cong \begin{cases} 0 & ; k < n \\ p_n(\rho)^{k-n} & ; k \geq n \end{cases} \tag{10}$$

$$\sum_{k=0}^{\infty} p_k \cong \sum_{k=n}^{\infty} p_k = 1$$

$$\sum_{k=n}^{\infty} p_n(\rho)^{k-n} \cong 1$$

$$p_n \sum_{k=n}^{\infty} \rho^{k-n} \cong 1$$

$$p_n \frac{1}{1-\rho} \cong 1$$

$$p_n \cong 1-\rho = 1-\frac{\lambda}{\mu_1+\cdots+\mu_n}$$

$$p_k \cong \rho^{k-n}\,(1-\rho) \tag{11}$$

3.3. Finding performance measures

The mean number of customers in queue is,

$$L_q = E(\xi_q) \cong \sum_{k=n}^{\infty} (k-n)p_k \tag{12}$$

putting the value of p_k in equation (12) we have,

$$L_q = E(\xi_q) \cong \sum_{k=n}^{\infty} (k-n)(1-\rho)(\rho)^{k-n} = \frac{\lambda}{c-\lambda}$$

In $M/\vec{M}/n$ system the number of customers in queue has geometric distrubition with parameter $p = (1-\rho)$, hence mean waiting time in queue is:

$$L_q \cong \frac{q}{p} = \frac{\frac{\lambda}{c}}{1-\frac{\lambda}{c}} = \frac{\lambda}{c-\lambda} \tag{13}$$

Let η be the service time. Then mean waiting time in the system is obtained as,

$$w = w + \eta$$

$$E(w) = E(w_q) + E(\eta) = \frac{1}{(c-\lambda)} + \frac{1}{\frac{\mu_1+\cdots\mu_n}{n}} = \frac{1}{(c-\lambda)} + \frac{n}{c} \tag{15}$$

4. Results

If the heterogeneous system is working in heavy tarffic then $E(\xi), E(\xi_q), E(w), E(w_q)$ means are fixed and do not change under condition $\mu_1 + \mu_2 + \cdots + \mu_n = c$. This result shows that there is no differences in heavy working queueing systems with n service channels wheter each service channel has the same service time or not. If $c = n\mu$ taken then both queue systems have the same performance measures. As a result, making a heavy working parallel system homogeneous does not increase the efficiency of the system but increasing total service capacity makes the system works faster.

Table 1: Performance measures of the $M / \vec{M} / n$ queueing systems.

Queue System	$M/\vec{M}/n$ System (Heterogeneous)	$M/\vec{M}/n$ System (Heterogeneous) Heavy traffic
p_k	$= \begin{cases} p_0 \dfrac{(n-k)!}{n!} E_k & ; k \le n \\ \dfrac{\rho_1 \dots \rho_n}{n!} \left(\dfrac{\lambda}{c}\right)^{k-n} p_0 & ; k > n \end{cases}$	$\cong \begin{cases} 0 & ; k < n \\ p_n (\rho)^{k-n} & ; k \ge n \end{cases}$
p_0	$= \dfrac{1}{\dfrac{1}{n!}\left[\sum_{k=0}^{n-1}(n-k)!\, E_k + \dfrac{E_n}{1-\dfrac{\lambda}{c}}\right]}$	$p_n \cong 1 - \rho$
L_q	$p_w \cdot \dfrac{\alpha}{1-\alpha} = p_w \cdot \dfrac{\lambda}{c-\lambda}$	$\cong \dfrac{\rho}{1-\rho} = \dfrac{\lambda}{c-\lambda}$
$E(W_q)$	$\dfrac{p_w}{(c-\lambda)}$	$\cong \dfrac{1}{(c-\lambda)}$
$E(W)$	$\dfrac{p_w}{(c-\lambda)} + \dfrac{n}{c}$	$\cong \dfrac{1}{(c-\lambda)} + \dfrac{n}{c}$

As seen in Table 1, the probability p_w in $M/\vec{M}/n$ queue system tends to 1 when the traffic density of the system becomes heavier.

References

[1] Everitt, D.E., Downs, T., 1983, The Output of the M / M / s Queue, operations Research, Vol.32, p. 796-806

[2] Gumbel, M., 1960, Waiting Lines With Heterogeneous Servers, Operations Researches, 8(4).

[3] Mitrinovic, D. S., 1970, Analytic Inequalities, New York.

[4] Pala, RE., Beloso, A.C., Hines W. W., Waiting Line Models An Introduction to Their Theory and Application, Reinhold Publishing Corporation, New York, 1966.

[5] Seth, K., 1976, Optimal Service Policies, Just after Idle Periods, on Two-Server Heterogeneous

Queueing Systems, Operations Research, Vol.25, p.357-360

[6] Shahbazov, A. A., 1984, The Loss Queue With Different Servers, Zastosav Mat., 18(2).

[7] Shahbazov, A. A., 1996, An Analysis And Optimization Of Stochastic Service Systems With Heterogeneous Channel And Poisson Arrival, Pure And Applied Mathematatika Sciences, 1(2).

[8] Singh, P.V., 1968, Two-Server Markovian Queues with Balking: Heterogeneous vs. Homogeneous Server, Case Western Reserve University, Technical Memorandum No.113.

[9] Stidham, S., 1976, On The Optimality Of Single Server Quening System, Operation Researches, 18(4).