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The Simulation and Control of Traffic Intensity in Hypoexponential and Coxian Queueing Systems with a Method Based on Sequential Probability Ratio Tests

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Abstract

The objective of the control technique is to detect changes in traffic intensity ρ of a queueing system as quickly as possible, then take appropriate corrective actions, and determine how much of a sample size is needed in the applications. Thus, the sequential probability ratio test provides a saving of up to fifty per cent in the sample size according to traditional methods. Furthermore, the use of SPRT is easy for observing only the number of customers in the system at successive departure periods Q_n, which are embedded Markov points. This paper gives a method on the control of traffic intensity (ρ) of Hypoexponential and Coxian queueing systems. This method uses The Sequential Probability Ratio Test (SPRT) based on the number of arrivals X_n during the n^{th} service period. Two theorems are given on the subject and these theorems are proved. Numerical illustrations for each model are graphically given by using Matlab software.

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1. Introduction

The theory of SPRT for a sequence of observations forming a finite Markov chain is given in [1]. After, based upon the theory of [1] is discussed statistical quality control and SPRT procedures for the control of traffic intensity in [2–9]. This method aims to detect changes in traffic intensity by observing only the number of customers in the system at successive departure periods Q_n, which are embedded Markov points. Recently, a SPRT to regulate the traffic intensity based on the number of arrivals during the n^{th} service periods for $M/E_k/1$ queue is proposed in [10] and an autoregressive process based on the number of customers at the departure point and its application to the queueing model are given in [11]. A similar topic has been studied for the Hyperexponential and the mixed Erlang queueing systems previously at [14]. This study consists of two parts. In the first part, we give two theorems on obtaining the probability density function of a hypoexponential and a Coxian queues respectively. These theorems are proved. In the second part of the application, the queue lengths have been generated randomly for different arrival rates (λ), different service rates (μ) and selected system capacities (N) by using Matlab software. Traffic intensities have also been calculated. With the obtained data and predetermined α and β, a simple hypotheses has been established. Accepting or rejecting the hypothesis has been examined by SPRT. After, the largest latent root $\lambda(t,\rho)$ of the $P(t)$ matrix has been computed by fixing the values for t and ρ. Their graphs have been drawn. It has been found that there exists one and only one real $t_0 \neq 0$ such that $\lambda_0(t_0) = 1$. The OC and ASN have been calculated with obtained values and graphically shown.

2. SPRT method in Queueing Theory

Consider the single server queue where arrivals occur according to a Poisson process with rate λ per unit time. The service times of customers are independent and identically distributed random variables with the distribution$B(x)$. For this system, the queue lenghts at service completion points form an imbedded Markov chain. Q_n will be the number of customers left behind by the nth departing customer. The capacity of the queueing system restricted to N. Then the transition probability matrix of the imbedded chain is given as,

$$P=\{P_{ij}(\rho)\}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ N\text{-}1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \cdots & N\text{-}1 \end{array} \\ \begin{bmatrix} k_0 & k_1 & k_2 & \cdots & 1-\sum_0^{k-2} k_j \\ k_0 & k_1 & k_2 & \cdots & 1-\sum_0^{k-2} k_j \\ & k_0 & k_1 & \cdots & 1-\sum_0^{k-3} k_j \\ & & & & \cdot \\ & & & & \cdot \\ & & & & \cdot \\ & & & & 1-k_0 \end{bmatrix} \end{array}$$

where

$$k_n = P\{n \text{ arrivals during a service period}\} = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^j}{j!}dB(t).$$

Consider the sequence of observations, $Q_0, Q_1, Q_2, \dots, Q_n$. The joint probability of observing this sequence under H_0 and H_1 is given by

$$\Pr\{Q_0, Q_1, Q_2, \dots, Q_a; \rho_i\} = P(Q_0; \rho_i)\prod_{j=1}^{n} P(Q_j \backslash Q_{j-1};\ \rho_i), \quad i = 0,1.$$

Then the likelihood ratio

$$L = \frac{P(Q_0;\rho_1)\prod_{j=1}^{n} P(Q_j \backslash Q_{j-1};\ \rho_1)}{P(Q_0;\rho_0)\prod_{j=1}^{n} P(Q_j \backslash Q_{j-1};\ \rho_0)} \tag{1}$$

$$Z_0 = ln\frac{P(Q_0;\rho_1)}{P(Q_0;\rho_0)} \tag{2}$$

$$Z_r = ln\frac{P(Q_r \backslash Q_{r-1};\rho_1)}{P(Q_r \backslash Q_{r-1};\rho_0)}, \quad (r \geq 1) \tag{3}$$

Let $A = (1-\beta)/\alpha$ and $B = \beta/(1-\alpha)$, where α and β are the probabilities of the errors of the first and second type. Then, Wald's SPRT [13] to test $H_0 : \rho = \rho_0$ aganist $H_1 : \rho = \rho_1$ becomes: Observe $\{Q_i\}$ $(i = 0,1,2, \dots)$ successively and at stage $n \geq 1$,

1) accept H_0 if $\sum_0^n Z_r \leq lnB$,
2) accept H_1 if $\sum_0^n Z_r \geq lnA$,
3) continue by observing Q_{n+1} if $\ln B < \sum_0^n Z_r < \ln A$.

If we assume $Q_0 = i_0$ is specified and denote by n_{ij} the number of transitions $i \rightarrow j$ up to and including the n^{th} transition, then the likelihood ratio given in (1) reduces to:

$$L = \prod_{i,j} P_{ij}^{n_{ij}}(\rho_1) \,/ \prod_{i,j} P_{ij}^{n_{ij}}(\rho_0) \tag{4}$$

and

$$ln\, L = \sum_{i,j} n_{ij} ln\frac{P_{ij}(\rho_1)}{P_{ij}(\rho_0)} \tag{5}$$

Then the SPRT for testing the hypothesis $H_0 : \rho = \rho_0$ against $H_1 : \rho = \rho_1$, will have its continuation region

$$ln\,B < \sum_{i,j} n_{ij} ln \frac{P_{ij}(\rho_1)}{P_{ij}(\rho_0)} < ln\,A \tag{6}$$

we will show in the sequel that the logarithm of the likelihood ratio can be written in the form,

$$ln\,L = an + \sum_{i,j} n_{ij} c_{ij} \tag{7}$$

where a and c_{ij}, and P_{ij} are constants depending upon the parameters ρ_1, ρ_0 and the transition probabilities P_{ij}. Thus (6) reduces to:

$$ln\,B - an < \sum_{i,j} n_{ij} c_{ij} < lnA - an \tag{8}$$

3. Operating Characteristic and Average Sample Number

Approximate formulas for the OC and ASN functions are given in [1]. A numerical search technique is employed to find OC and ASN by [6]. When the stated space of $\{Q_n\}$ is finite, the OC function for the SPRT can be obtained as:

$$L(\rho) \cong \frac{A^{t_0(\rho)} - 1}{A^{t_0(\rho)} - B^{t_0(\rho)}}, \qquad \text{if } t_0(\rho) \neq 0 \tag{9}$$

$$\cong \frac{\ln A}{ln\,A - lnB}, \qquad \text{if } t_0(\rho) = 0 \tag{10}$$

where $t_0(\rho)$ is the non-zero real root of the equation $\lambda_0(t, \rho) = 1$. The ASN can then be obtained as:

$$E(n; \rho) \cong \frac{L(\rho) lnB + \{1 - L(\rho)\} lnA}{\lambda_0'(0)}, \qquad \text{if } \lambda_0' \neq 0 \tag{11}$$

$$\cong \frac{L(\rho)\{lnB\}^2 + \{1 - L(\rho)\}\{lnA\}^2}{\lambda_0''}, \qquad \text{if } \lambda_0' = 0 \tag{12}$$

where,

$$\lambda_0'(t, \rho) \cong \frac{1}{12h}\{\lambda_{-2} - 8\lambda_{-1} + 8\lambda_1 - \lambda_2\} \tag{13}$$

$$\lambda_0''(t, \rho) \cong \frac{1}{12h^2}\{-\lambda_{-2} + 16\lambda_{-1} - 30\lambda_0 + 16h_1 - \lambda_2\} \tag{14}$$

4. SPRT for Phase Type Distribution

The exponential distribution is very widely used in performance modelling. The reason, of course, is that mathematical tractability flows from the memoryless property of this distribution. But sometimes mathematical tractability is not sufficient to overcome the need for a model process in which the exponential distribution is simply not adequate. This leads us to explore ways in which we can develop more general distributions while maintaining some of the tractability of the exponential. This is precisely what phase-type distributions permit us to do [12].

4.1. Hypoexponential Queue

In probability theory the hypoexponential distribution is a continuous distribution, that has found use in the same fields as Erlang distribution, such as queueing theory, teletraffic engineering and more generally in stochastic processes. The Erlang distribution is a series of k exponential distributions all with rate μ. The hypoexponential is a series of k exponential distributions each with their own rate μ_i, the rate of the i^{th} exponential distribution. Once again, only one customer can be in the process of receiving service at any one time, i.e., both phases cannot be active at the same time [12].

Theorem 1. The density function of the service time is given by:

$$\frac{dB(x)}{dx} = \frac{\mu_1\mu_2}{\mu_1-\mu_2}(e^{-\mu_2 x} - e^{-\mu_1 x}), \qquad (x > 0)$$

where $k = 2$ is an integer. In this case, we have

$$k_n = \frac{1}{\mu_1-\mu_2}\left[\mu_1\left(\frac{\rho_2}{\rho_2+1}\right)^n\left(\frac{1}{\rho_2+1}\right) - \mu_2\left(\frac{\rho_1}{\rho_1+1}\right)^n\left(\frac{1}{\rho_1+1}\right)\right] \tag{15}$$

where $k_n = P\{$n arrivals during a service period$\}$

$$= \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^j}{j!}dB(t)$$

Proof.

$$k_n = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!}\frac{\mu_1\mu_2}{\mu_1-\mu_2}(e^{-\mu_2 t} - e^{-\mu_1 t})dt$$

$$= \frac{\mu_1\mu_2}{\mu_1-\mu_2}\frac{\lambda^n}{n!}\left[\int_0^\infty e^{-t(\lambda+\mu_2)}t^n dt - e^{-t(\lambda+\mu_1)}t^n dt\right]$$

$$= \frac{\mu_1\mu_2}{\mu_1-\mu_2}\frac{\lambda^n n!}{n!}\left[\left(\frac{1}{\lambda+\mu_2}\right)^{n+1} - \left(\frac{1}{\lambda+\mu_1}\right)^{n+1}\right]$$

$$= \frac{1}{\mu_1-\mu_2}\left[\mu_1\left(\frac{\rho_2}{\rho_2+1}\right)^n\left(\frac{1}{\rho_2+1}\right) - \mu_2\left(\frac{\rho_1}{\rho_1+1}\right)^n\left(\frac{1}{\rho_1+1}\right)\right]$$

in which,

$$P_{ij}(\rho) = k_{j-i+1}, \quad i = 1,2,\dots,N;\ j = 0,1,\dots,N-1;\ j \geq i-1$$
$$P_{0j}(\rho) = k_j, \quad j = 0,1,\dots,N-1$$
$$P_{0N}(\rho) = 1 - \sum\nolimits_{0}^{N-1} k_n$$

and

$$P_{iN}(\rho) = 1 - \sum\nolimits_{0}^{N-i} k_n, \qquad i = 1,2,\dots,N$$
$$P_{iN}(\rho) = 1 - \left(\sum\nolimits_{r=0}^{N-i} \frac{1}{\mu_1 - \mu_2}\left[\mu_1\left(\frac{\rho_2}{\rho_2+1}\right)^r\left(\frac{1}{\rho_2+1}\right) - \mu_2\left(\frac{\rho_1}{\rho_1+1}\right)^r\left(\frac{1}{\rho_1+1}\right)\right]\right)$$

With these values of $P_{ij}(\rho)$, the logarithm of the likelihood ratio will be

$$ln\, L = \sum_{i,j} n_{ij}\, ln \frac{P_{ij}(\rho_1)}{P_{ij}(\rho_0)} = an + \sum_{i,j} n_{ij} c_{ij}$$

where

$$ln\, L = \sum_{i,j} n_{ij}\, ln \left[\frac{\frac{1}{\mu_{1(1)} - \mu_{2(1)}}\left(\mu_{1(1)}\left(\frac{\rho_{2(1)}}{\rho_{2(1)}+1}\right)^n\left(\frac{1}{\rho_{2(1)}+1}\right) - \mu_{2(1)}\left(\frac{\rho_{1(1)}}{\rho_{1(1)}+1}\right)^n\left(\frac{1}{\rho_{1(1)}+1}\right)\right)}{\frac{1}{\mu_{1(0)} - \mu_{2(0)}}\left(\mu_{1(0)}\left(\frac{\rho_{2(0)}}{\rho_{2(0)}+1}\right)^n\left(\frac{1}{\rho_{2(0)}+1}\right) - \mu_{2(1)}\left(\frac{\rho_{1(0)}}{\rho_{1(0)}+1}\right)^n\left(\frac{1}{\rho_{1(0)}+1}\right)\right)}\right]$$

$$= \left[\frac{\mu_{1(0)} - \mu_{2(0)}}{\mu_{1(1)} - \mu_{2(1)}}\right] + \sum_{i,j} n_{ij} ln \left[\frac{\left(\mu_{1(1)}\left(\frac{\rho_{2(1)}}{\rho_{2(1)}+1}\right)^n\left(\frac{1}{\rho_{2(1)}+1}\right) - \mu_{2(1)}\left(\frac{\rho_{1(1)}}{\rho_{1(1)}+1}\right)^n\left(\frac{1}{\rho_{1(1)}+1}\right)\right)}{\left(\mu_{1(0)}\left(\frac{\rho_{2(0)}}{\rho_{2(0)}+1}\right)^n\left(\frac{1}{\rho_{2(0)}+1}\right) - \mu_{2(0)}\left(\frac{\rho_{1(0)}}{\rho_{1(0)}+1}\right)^n\left(\frac{1}{\rho_{1(0)}+1}\right)\right)}\right] \qquad (16)$$

and

$$c_{ij}$$
$$= ln \left[\frac{\left(\mu_{1(1)}\left(\frac{\rho_{2(1)}}{\rho_{2(1)}+1}\right)^{j-i+1}\left(\frac{1}{\rho_{2(1)}+1}\right) - \mu_{2(1)}\left(\frac{\rho_{1(1)}}{\rho_{1(1)}+1}\right)^{j-i+1}\left(\frac{1}{\rho_{1(1)}+1}\right)\right)}{\left(\mu_{1(0)}\left(\frac{\rho_{2(0)}}{\rho_{2(0)}+1}\right)^{j-i+1}\left(\frac{1}{\rho_{2(0)}+1}\right) - \mu_{2(0)}\left(\frac{\rho_{1(0)}}{\rho_{1(0)}+1}\right)^{j-i+1}\left(\frac{1}{\rho_{1(0)}+1}\right)\right)}\right] \qquad (17)$$

$$i = 1,2,\dots,N;\ j = 0,1,\dots,N-1;\ j \geq i-1$$

$$c_{0j} = ln\left[\frac{\left(\mu_{1(1)}\left(\frac{\rho_{2(1)}}{\rho_{2(1)}+1}\right)^{j}\left(\frac{1}{\rho_{2(1)}+1}\right)-\mu_{2(1)}\left(\frac{\rho_{1(1)}}{\rho_{1(1)}+1}\right)^{j}\left(\frac{1}{\rho_{1(1)}+1}\right)\right)}{\left(\mu_{1(0)}\left(\frac{\rho_{2(0)}}{\rho_{2(0)}+1}\right)^{j}\left(\frac{1}{\rho_{2(0)}+1}\right)-\mu_{2(0)}\left(\frac{\rho_{1(0)}}{\rho_{1(0)}+1}\right)^{j}\left(\frac{1}{\rho_{1(0)}+1}\right)\right)}\right] \tag{18}$$

$j = 0,1,\dots,N-1$

$$c_{0j} = ln\left[\frac{1-\left(\sum_{r=0}^{N-i}\mu_{1(1)}\left(\frac{\rho_{2(1)}}{\rho_{2(1)}+1}\right)^{r}\left(\frac{1}{\rho_{2(1)}+1}\right)-\mu_{2(1)}\left(\frac{\rho_{1(1)}}{\rho_{1(1)}+1}\right)^{r}\left(\frac{1}{\rho_{1(1)}+1}\right)\right)}{1-\left(\sum_{r=0}^{N-i}\mu_{1(0)}\left(\frac{\rho_{2(0)}}{\rho_{2(0)}+1}\right)^{r}\left(\frac{1}{\rho_{2(0)}+1}\right)-\mu_{2(0)}\left(\frac{\rho_{1(0)}}{\rho_{1(0)}+1}\right)^{r}\left(\frac{1}{\rho_{1(0)}+1}\right)\right)}\right] \tag{19}$$

$i = 1,2,\dots,N$

and

$c_{0N} = c_{1N}$

4.2. Coxian Queue

Coxian distributions imagine prominently in the theory of networks of queues. Their importance is in large part due to their universality: any distribution function can be The Coxian distribution is a generalization of hypoexponential distribution. Instead of only being able to enter the absorbing state from state k it can be reached from any phase.

Theorem 2. The density function of the service time in a Coxian queue is given as:

$$\frac{dB(x)}{dx} = \sigma e^{Sx} S^0, \quad (x > 0)$$

where σ_{nx1} is a row vector, S_{nxn} is a square matrix and S^0_{1xn} is a column vector. In this case, we have:

$$k_n = \sigma\left(\frac{\rho}{\rho I - \frac{S}{\mu}}\right)^{n}\left(\frac{1}{\lambda I - S}\right) S^0 \tag{20}$$

where $k_n = P$\{n arrivals during a service period\}

$$= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^j}{j!} dB(t).$$

Proof.

$$k_n = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!} \sigma e^{Sx} S^0 \, d(t)$$

$$= \int_0^\infty \frac{\lambda^n t^n}{n!} \sigma e^{-t(\lambda I - S)} S^0 d(t)$$

$$= \frac{\lambda^n}{n!} \sigma \left\{ \int_0^\infty e^{-t(\lambda I - S)} t^n \, d(t) \right\} S^0$$

$$= \frac{\lambda^n}{n!} \sigma \frac{1}{(\lambda I - S)^{n+1}} n! \, S^0$$

$$= \lambda^n \sigma \frac{1}{(\lambda I - S)^{n+1}} S^0$$

$$= \sigma \left(\frac{\lambda}{\lambda I - S} \right)^n \left(\frac{1}{\lambda I - S} \right) S^0$$

If we take $D = S/\mu$ in the last equation above, then it is rewritten as following:

$$= \sigma \left(\frac{\rho}{\rho I - D} \right)^n \left(\frac{1}{\lambda I - S} \right) S^0$$

where,

$$P_{ij}(\rho) = k_{j-i+1}, \quad i = 1,2, \dots, N; \; j = 0,1, \dots, N-1; \; j \geq i-1$$

$$P_{0N}(\rho) = 1 - \sum\nolimits_0^{N-1} k_n$$

and

$$P_{iN}(\rho) = 1 - \textstyle\sum_0^{N-i} k_n, \qquad i = 1,2, \dots, N$$

$$P_{iN}(\rho) = 1 - \sum_{r=0}^{N-i} \sigma(\rho(\rho I - D)^{-1})^r (\lambda I - S)^{-1} S^0.$$

With these values for $P_{ij}(\rho)$, the logarithm of the likelihood ratio will be as following:

$$\ln L = \sum_{i,j} n_{ij} \ln \frac{P_{ij}(\rho_1)}{P_{ij}(\rho_0)} = an + \sum_{i,j} n_{ij} c_{ij}.$$

Where,

$$ln\,L = \sum_{i,j} n_{ij} ln \left(\frac{\sigma_1 \left(\frac{\rho_1}{\rho_1 I - D}\right)^n \left(\frac{1}{\lambda_1 I - S}\right) S_1^0}{\sigma_0 \left(\frac{\rho_0}{\rho_0 I - D}\right)^n \left(\frac{1}{\lambda_0 I - S}\right) S_0^0} \right)$$

$$ln\,L = \sum_{i,j} n_{ij} ln \left(\frac{\sigma_1(\rho_1(\rho_1 I - D_1)^{-1})^n (\lambda_1 I - S_1)^{-1} S_1^0}{\sigma_0(\rho_0(\rho_0 I - D_0)^{-1})^n (\lambda_0 I - S_0)^{-1} S_0^0} \right) \tag{21}$$

and

$$c_{ij=} ln \left(\frac{\sigma_1(\rho_1(\rho_1 I - D_1)^{-1})^{j-i+1} (\lambda_1 I - S_1)^{-1} S_1^0}{\sigma_0(\rho_0(\rho_0 I - D_0)^{-1})^{j-i+1} (\lambda_0 I - S_0)^{-1} S_0^0} \right) \tag{22}$$

$i = 1,2, \dots, N;\ j = 0,1, \dots, N-1;\ j \geq i-1$

$$c_{0j} = ln \left(\frac{\sigma_1(\rho_1(\rho_1 I - D_1)^{-1})^{j} (\lambda_1 I - S_1)^{-1} S_1^0}{\sigma_0(\rho_0(\rho_0 I - D_0)^{-1})^{j} (\lambda_0 I - S_0)^{-1} S_0^0} \right) \tag{23}$$

$j = 0,1, \dots, N-1$

$$c_{iN} = ln \left(\frac{1 - \sum_{r=0}^{N-i} \sigma_1(\rho_1(\rho_1 I - D_1)^{-1})^{r} (\lambda_1 I - S_1)^{-1} S_1^0}{1 - \sum_{r=0}^{N-i} \sigma_0(\rho_0(\rho_0 I - D_0)^{-1})^{r} (\lambda_0 I - S_0)^{-1} S_0^0} \right) \tag{24}$$

$i = 1,2, \dots, N$

$c_{0N} = c_{1N}.$

5. Illustrative Examples

In the application part, the queue lengths are generated randomly for different arrival rates (λ), different service rates (μ) and selected system capacities (N) by using MATLAB 7.10.0 (R2010a) programming. Traffic intensities are also calculated. With the obtained data and predetermined α and β, a simple hypotheses are established. Accepting or rejecting the hypothesis are decided by SPRT. After, the largest latent root $\lambda(t, \rho)$ of the $P(t)$ matrix is computed by fixing the values for t and ρ. Their graphs are drawn. It is found that there exists one and only one real $t_0 \neq 0$ such that $\lambda_0(t_0) = 1$. The OC and ASN calculated with obtained values and their graphs are drawn by Microsoft Office Excel 2007 programming.

Example 1: Consider a $M/HYPO_2/1$ queue (with poisson arrivals, two phase-type hypoexponential service, fixed N). Let $\alpha_1 = 0.4$ and $\alpha_2 = 0.6$ be the probability for the upper phase and the probability for the lower phase, and $\alpha = 0.05$ and $\beta = 0.10$ be the first and second type of errors, respectively. Let $N = 7$ be the capacity of the queuing system. The mean value of ρ is calculated using the following formula,

$$\rho = \frac{\lambda(\mu_1 + \mu_2)}{\mu_1 \mu_2} \tag{25}$$

Suppose we wish to maintain ρ at level 0.1 and wish to detect whether its value has increased. Then, the hypothesis test is $H_0: \rho_0 = 0.1$ against alternative $H_1: \rho_1 = 0.04$.

Let $t_0 = 0, t_1, t_2, t_3, \dots \dots ..$ be a discrete set of the number of customers remaining at points of departure in the $M/HYPO_2/1$ queue. The number of customers remaining at the 100 points departure is given as the following:

3 2 1 0 4 4 4 3 3 2 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1 2 2 1 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 1 1 0 0 0 2 1 0 1 1 1 1 0 4 3 2 1 4 3 2 1 2 1 1 0 0 0 1 0 0 0 0 0 1 2 2 2 1 1 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0

The state space is $E = \{0,1,2,3,4,5,6,7\}$. Table 1 shows n_{ij} that the number of transitions $i \to j$

Table 1: The number of transitions $i \to j$

i \ j	0	1	2	3	4	5	6	
0	41	9	1	0	2	0	0	53
1	13	8	3	0	1	0	0	25
2	0	8	3	0	0	0	0	11
3	0	0	4	1	0	0	0	5
4	0	0	0	3	2	0	0	5
5	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0
							$n =$	99

and Table 2 shows the values of c_{ij} calculating for SPRT.

Table 2: The values of c_{ij}

i \ j	0	1	2	3	4	5	6
0	0.76	0.24	-0.26	-0.75	-1.23	-1.70	-0.06
1	0.76	0.24	-0.26	-0.75	-1.23	-1.70	-0.06
2	0	0.76	0.24	-0.26	-0.75	-1.23	-0.09
3	0	0	0.76	0.24	-0.26	-0.75	-0.12
4	0	0	0	0.76	0.24	-0.26	-0.15
5	0	0	0	0	0.76	0.24	-0.16
6	0	0	0	0	0	0.76	-0.12

$$ln\, L = \sum_{i,j} n_{ij}c_{ij} = 53.13$$

$$ln\, B < \sum_{i,j} n_{ij}c_{ij} < lnA$$

$$-2.25 < \sum_{i,j} n_{ij}c_{ij} < 2.89$$

From $ln\, B < \sum_{i,j} n_{ij}c_{ij} < lnA$, the decision region , $\sum_{i,j} n_{ij}c_{ij}$, is small lnA $(53.13 > 2.89.)$ Therefore, the hypothesis H_0 is rejected.

For the computation of the OC and ASN of the SPRT, the largest latent root $\lambda_0(t,\rho)$ of the $P(t)$ matrix is computed by fixing the values for t and ρ. Figure 3 shows the graphs of $\lambda_0(t,\rho)$ which are plotted for the values of t for different values of ρ.

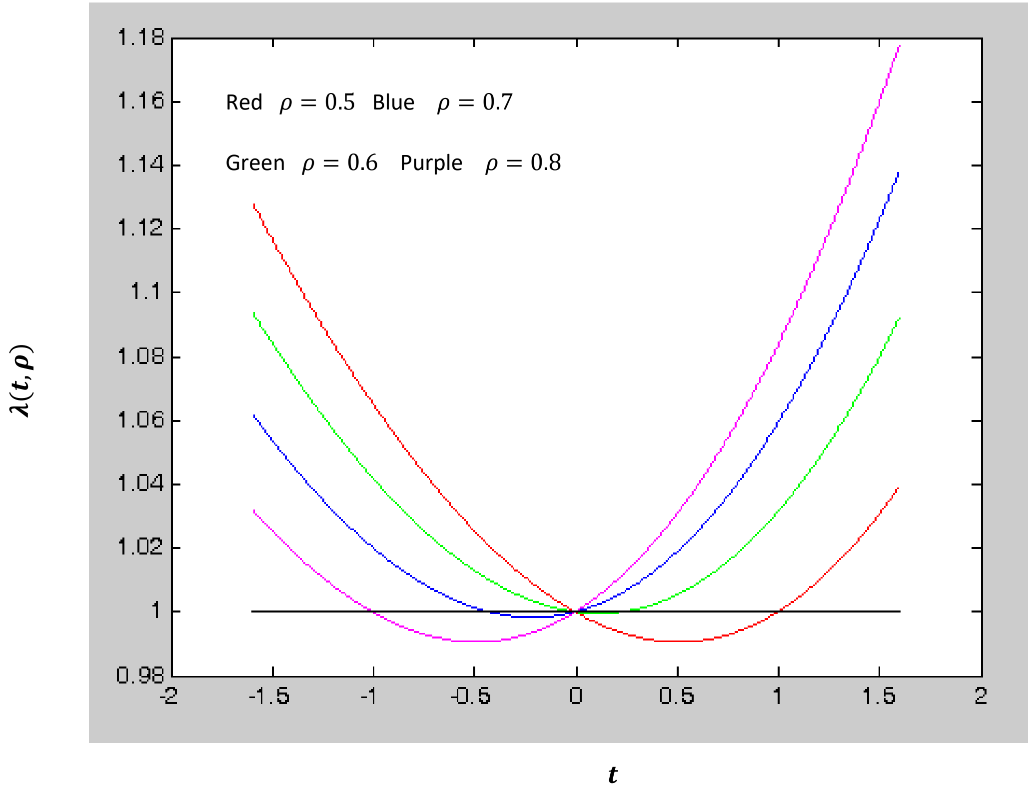

Figure 3: Graph of $\lambda(t,\rho)$ aganist t for testing $H_0\!: \rho_0 = 0.5, H_1\!: \rho_1 = 0.8$

As can be seen in Figure 3, there exists one and only one real $t_0 \neq 0$ such that $\lambda_0(t_0) = 1$. The derivative of $\lambda(t,\rho)$ at $t = 0$ is computed. The OC and ASN functions are then evaluated using the expressions between (9) and (14). The results of the OC and ASN functions for testing $H_0\!: \rho_0 = 0.04$ aganist $H_1\!: \rho_1 = 0.09$ are given in Table 4 .

Table 3 shows the computation of the OC and ASN values.

Table 3: The OC and ASN values

ρ	$t(\rho)$	$\lambda_0'(0,\rho)$	$L(\rho)$	$E(n,\rho)$
0.1	1	-0.0299	0.95	66.73
0.09	0.75	-0.0213	0.90	82.34
0.08	0.45	-0.0127	0.82	102.37
0.07	0.18	0.0041	0.67	136.10
0.06	0.20	0.0045	0.44	144.10
0.05	0.57	0.0131	0.24	127.18
0.04	-1	0.0217	0.10	109.76

Example 2: Consider a $M/C_2/1$ queue (with poisson arrivals, two phase-type Coxian service, fixed N). Let $\alpha_1 = 0.3$ and $\alpha_2 = 0.7$ be the probability for the upper phase and the probability for the lower phase, and $\alpha = 0.1$ and $\beta = 0.05$ be the first and second type of errors, respectively. Let $N = 3$ be the capacity of the queuing system. The mean value of ρ is calculated using the following formula,

$$\rho = \frac{\lambda(\mu_2 + \mu_1\alpha_1)}{\mu_1\mu_2} \qquad (26)$$

Suppose we wish to maintain ρ at the level 0.04 and we wish to detect whether its value is increased. Then, the hypothesis test is $H_0: \rho_0 = 0.04$ aganist alternative $H_1: \rho_1 = 0.09$.

For $\rho_0 = 0.04$;

$\sigma_{2x1} = [1\ 0]$

$S_{2x2} = \begin{bmatrix} 0.1 & 0.2 \\ 0.5 & 0.6 \end{bmatrix}$

$S^0_{1x2} = [\ 0.3\ \ 0.6]$

$I_{2x2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

For $\rho_1 = 0.09$;

$\sigma_{2x1} = [1\ 0]$

$S_{2x2} = \begin{bmatrix} -1 & 0.3 \\ 0 & -2 \end{bmatrix}$

$S^0_{1x2} = [\ 0.7\ \ 0.6]$

$I_{2x2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Let $t_0 = 0, t_1, t_2, t_3, \ldots$ be a discrete set of the number of customers remaining at points of departure in the $M/C_2/1$ queue. The number of customers remaining at the 20 points departure is given as the following,

1 0 1 0 0 0 0 2 2 1 0 2 1 2 1 0 2 1 1 0

State space is $E = \{0,1,2,3\}$. Table 4 shows n_{ij} that the number of transitions $i \to j$

Table 4: The number of transitions $i \to j$

i \ j	0	1	2	
0	3	1	3	7
1	5	1	1	7
2	0	4	1	5
			n	19

and Table 5 shows the values of c_{ij} calculating for SPRT.

Table 5: The values of c_{ij}

i \ j	0	1	2
0	0.87	-1.88	-0.14
1	0.87	-1.88	-0.14
2	0	0.87	-0.90

$$ln\, L = \sum_{i,j} n_{ij} c_{ij} = 5.19$$

$$-2.25 < \sum_{i,j} n_{ij}c_{ij} < 2.89$$

From $ln\, B < \sum_{i,j} n_{ij}c_{ij} < lnA$, the decision region , $\sum_{i,j} n_{ij}c_{ij}$, is smaller than lnA $(5.19 > 2.89)$ Therefore, the hypothesis H_0 is rejected.

For the computation of the OC and ASN of the SPRT, the largest latent root $\lambda_0(t,\rho)$ of the $P(t)$ matrix is computed by fixing the values for t and ρ. Figure 3 shows the graphs of $\lambda_0(t,\rho)$ which are plotted aganist the values of t for different values of ρ.

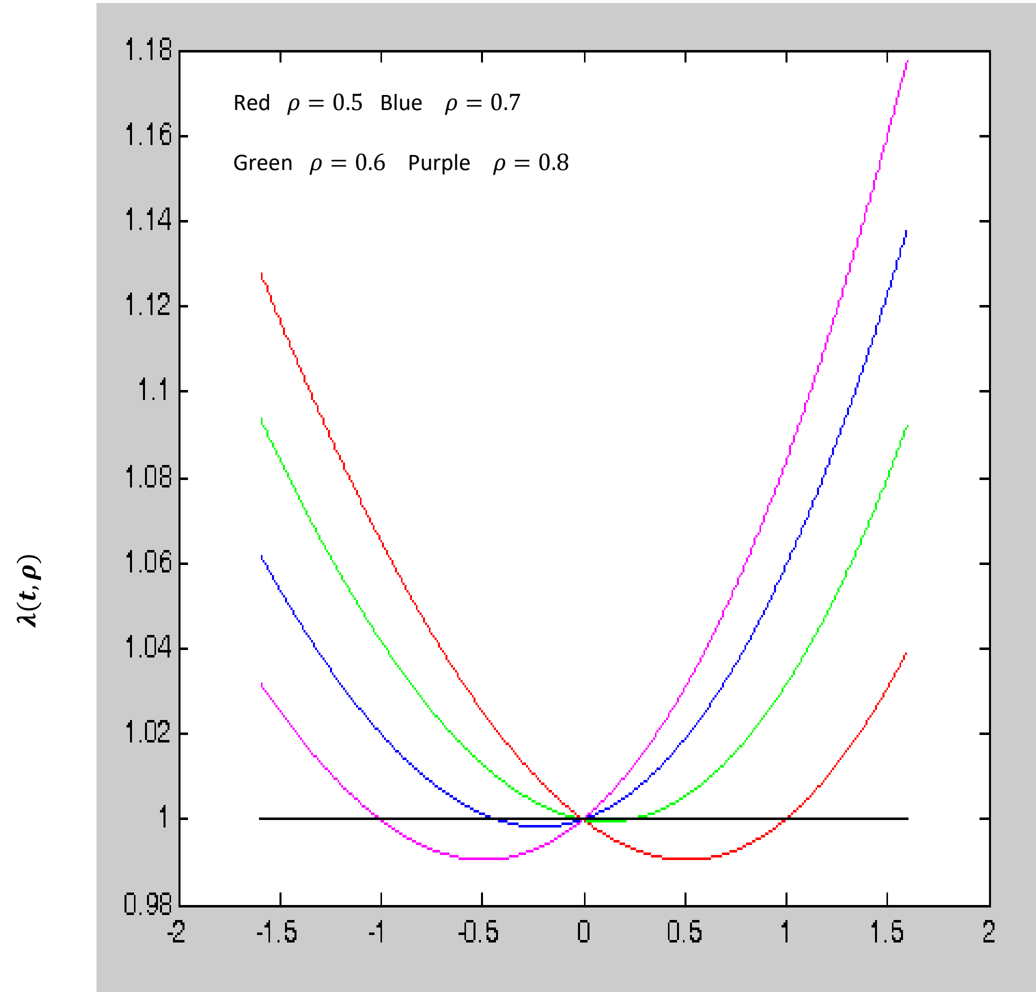

Figure 3: Graph of $\lambda(t,\rho)$ aganist t for testing $H_0\colon \rho_0 = 0.5$, $H_1\colon \rho_1 = 0.8$

As can be seen in Figure 3, there exists one and only one real $t_0 \neq 0$ such that $\lambda_0(t_0) = 1$. The derivative of $\lambda(t,\rho)$ at $t = 0$ is computed. The OC and ASN functions are then evaluated using the expressions between (9) and (14). The results of the OC and ASN functions for testing $H_0\colon \rho_0 = 0.04$ aganist $H_1\colon \rho_1 = 0.09$ are given in Table 6.

Table 6: The OC and ASN values

ρ	$t(\rho)$	$\lambda_0'(0,\rho)$	$L(\rho)$	$E(n,\rho)$
0.04	1	-0.0164	0.90	145.43
0.05	0.50	-0.0087	0.73	173.99
0.06	0.11	-0.0011	0.52	314.76
0.07	-0.34	0.0065	0.40	156.05
0.08	-0.67	0.0141	0.11	119.16
0.09	-1	0.0216	0.05	92.34

6. Conclusion

The objective of the control technique is to detect changes in the traffic intensity ρ as quickly as possible, then take appropriate corrective action and determine how much of a sample size is needed in the applications.

Here, the value of μ needed to calculate the traffic intensity is taken from the mean μ (25). If you recall, the purpose of a control technique is to detect changes in the traffic intensityρ. When ρ shifted to ρ_1 from the design level ρ_0 $(\rho_1 < \rho_0)$, is taken appropriate action to bring ρ_1 back to the design level ρ_0. In the same way, when ρ shifted to ρ_1 from the design level ρ_0 $(\rho_1 > \rho_0)$, is taken appropriate action to bring ρ_1 back to the design level ρ_0 [9]. Consequently, the control action could be to increase mean μ or decrease mean μ for the phase-type queueing systems.

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