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Comparative Study of Convergence of Sequence of Functions in a Banach Space

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Abstract

We discuss four types of convergence of sequence of functions in a Banach space. The types of convergence considered include pointwise, uniform, strong and weak convergence. It is shown that uniform convergence implies the pointwise convergence and the strong Convergence implied the weak convergence. We also show how basic analysis concepts are used in proving advanced concepts and also provide an alternative description of the exponential function e^x.

1. Introduction

First, we will analyze convergence of sequences of real numbers and the convergence of series in particular uniform convergence of the series. $M_n - test$ is used to prove uniform convergence of sequence of functions and end SECTION ONE by defining strong and weak convergence.

SECTION TWO is where we treat uniform convergence of series with integration and differentiation. An alternate description of the exponential function e^x is shown and it is used to prove that $\left(1+\frac{x}{n}\right)^n$ converges uniformly to e^x.

Keywords: Strong; weak; uniform; pointwise convergence.

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Finally in SECTION THREE we compare two different results: uniform convergence imply pointwise convergence but the reverse is not true and also strong convergence imply weak convergence although the reverse is not generally true except in the finite dimensional space.

Section One:

Preliminaries

Definition 1.1: A sequence a_n of real numbers is said to be convergent if there exists a real number a satisfying the condition: to every real number ε there corresponds a positive integer p such that $|a_n - a| < \varepsilon$ for all $n \geq p$. Under such circumstances we write $a_n \to a$ as $n \to \infty$.

Definition 1.2: If a_n is a sequence and n_k is a strictly increasing sequence $(n_1 < n_2 < n_3, \dots)$ of positive integers then a_{n_k} is called a subsequence of a_n.

Definition 1.3: If a_n is a convergent sequence of real numbers and $\lim_{n\to\infty} a_n = a$ then $a_{n_j} \to a$ as $j \to \infty$ for every subsequence a_{n_j} of a_n .

Remark: Given a positive real number ε choose a positive integer p such that $|a_n - a| < \varepsilon$ for all $n \geq p$. Then $\left|a_{n_j} - a\right| < \varepsilon$ for all $j \geq p$.

Let a_n be a complex number for every nonnegative integer n. Then $\sum_{n=0}^{\infty} a_n$ is called a series. For every positive integer k, let $S_k = \sum_{n=0}^{\infty} a_n = a_0 + a_1 + \cdots + a_k$. Then S_k is called the kth partial sum of $\sum_{n=0}^{\infty} a_n$.

The series $\sum_{n=0}^{\infty} a_n$ is said to be convergent if the sequence S_k is convergent. When the sequence S_k is convergent, let $A = \lim_{n\to\infty} S_k$. Then A is called the sum of $\sum_{n=0}^{\infty} a_n$ and we usually write $\sum_{n=0}^{\infty} a_n = A$.

The series $\sum_{n=0}^{\infty} a_n$ is said to be divergent if the sequence S_k is not convergent. The series $\sum_{n=0}^{\infty} a_n$ is said to be absolutely convergent if $\sum_{n=0}^{\infty} |a_n|$ is convergent.

Theorem 1.1: These two statements are equivalent:

(a) $\sum_{n=0}^{\infty} a_n$ is convergent

(b) Cauchy criterion: To every positive real number ε there corresponds a positive integer q such that $|\sum_{n=m}^{k} a_n| < \varepsilon$ for every pair of integers m, k satisfying $k \geq m \geq q$.

Remark: Let S_k be kth partial sum of $\sum_{n=0}^{\infty} a_n$ for every positive integer k. If (a) is true then S_k is convergent and so S_k is a Cauchy sequence of complex numbers. Given a positive real number ε choose a positive integer p such that $|S_k - S_j| < \varepsilon$ for every $j \geq p$ and $\geq p$. Let $q = p + 1$ then $|\sum_{n=m}^{k} a_n| = |S_k - S_{m-1}| < \varepsilon$ for every pair of integers m,k satisfying $k \geq m \geq q$. That proves $(a) \Longrightarrow (b)$

Similarly, if (b) is true then S_k is a Cauchy sequence of complex numbers and so S_k is convergent . That is equivalent to the assertion that $\sum_{n=0}^{\infty} a_n$ is convergent .Thus $(b) \Longrightarrow (a)$.

Corollary 1.1: If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent then it is convergent.

Remark: Given a positive real number ε choose a positive integer q such that $\sum_{n=m}^{\infty}|a_n| < \varepsilon$ for every pair of integers m, k satisfying $k \geq m \geq q$. Then $|\sum_{n=m}^{k} a_n| < \varepsilon$ for every pair of integers m, k satisfying $k \geq m \geq q$. Thus this result follows from Cauchy criterion.

Corollary 1.2: If $\sum_{n=0}^{\infty} a_n$ is convergent then $a_n \to 0$ as $n \to \infty$.

Remark: Given a positive real number ε choose a positive integer q such that $|\sum_{n=m}^{k} a_n| < \varepsilon$ .Then $|\sum_{n=m}^{k} a_n| < \varepsilon$ for every pair of integers m, k satisfying $k \geq m \geq q$. Then in particular $|a_n| < \varepsilon$ for all $n \geq q$. Thus $a_n \to 0$ as $n \to \infty$.

Definition 1.4: Let S be a nonempty set and $a_n = a_n(x)$ a sequence of real-valued functions on S. For every positive integer k let $s_k = \sum_{n=0}^{k} a_n(x)$ at every point $x \in S$. We say that $\sum_{n=0}^{\infty} a_n(x)$ is uniformly convergent on S if the sequence of functions s_k is uniformly convergent on S.

The **Cauchy criterion:**$\sum_{n=0}^{\infty} a_n(x)$ is uniformly convergent on S if and only if to every positive real number ε there corresponds a positive integer p such that $|\sum_{n=m}^{k} a_n(x)| < \varepsilon$ for all $x \in S$ whenever $k \geq m \geq p$.

Theorem 1.2(Weierstrass Test): Let τ_n be a sequence of nonnegative real numbers such that $\sum_{n=0}^{\infty} \tau_n$ is convergent. If S is a nonempty set and $a_n = a_n(x)$ a sequence of real-valued functions on S such that $|a_n(x)| \leq \tau_n$ for all $x \in S$ and $n \geq 0$ then $\sum_{n=0}^{\infty} a_n(x)$ is uniformly convergent on S.

Remark: Given a positive real number ε choose a positive integer p such that $\sum_{n=m}^{k} \tau_n < \varepsilon$ whenever $k \geq m \geq p$. Then $|\sum_{n=m}^{k} a_n(x)| \leq \sum_{n=m}^{k} |a_n(x)| \leq \sum_{n=m}^{k} \tau_n < \varepsilon$whenever $k \geq m \geq p$. Hence $\sum_{n=0}^{\infty} a_n(x)$ is uniformly convergent on S by the Cauchy criterion.

Definition 1.5: A series $\sum_{n=1}^{\infty} x_n$ converges in a normed space $(X, ||, ||)$ if the sequence of partial sums converges i.e. there exists $x \in X$ such that $||x_1 + x_2 + \cdots + x_n - x|| \to 0$ as $n \to \infty$.

In that case we write $\sum_{n=1}^{\infty} x_n = x$. If $\sum_{n=1}^{\infty} ||x_n|| < \infty$ then the series is called absolutely convergent [6].

Theorem 1.3: A normed space is complete if and only if every absolutely convergent series converges.

Remark: for a prove of this theorem refer to [6].

Hahn Banach Theorem 1.4: Let $(X, ||, ||)$ be a normed space and x_0 a nonzero element of X such that there exists a bounded linear functional $\tilde{f}$ on X such that $\tilde{f}(x_0) = ||x_0||$and $\| \tilde{f} \| = 1$.

Corollary 1.3: For every x in a normed space $(X, ||,||)$ we have $\| x \| = \text{Sup}_{f\epsilon X', f\neq 0} \frac{|f(x)|}{\|f\|}$. Hence if x_0 is such that $f(x_0) = 0$ for all $f \in X'$ then $x_0 = 0$.

Remarks: $\text{Sup}_{f\epsilon X', f\neq 0} \frac{|f(x)|}{\|f\|} \geq \frac{|\tilde{f}(x)|}{\|\tilde{f}\|} = |\tilde{f}| = \| x \|$, $\quad \text{Sup}_{f\epsilon X', f\neq 0} \frac{|f|}{\|f\|} \geq \| x \| \cdots (i)$

But $|f(x)| \leq \| f \| \| x \| \Longrightarrow \text{Sup}_{f\epsilon X', f\neq 0} \frac{|f|}{\|f\|} \leq \| x \| \cdots (ii)$ Combining (i) and (ii) we have

$\| x \| = \text{Sup}_{f\epsilon X', f\neq 0} \frac{|f(x)|}{\|f\|}$ which completes the proof.

Riesz's Representation Theorem 1.5: Every bounded linear functional f on a Hilbert space H can be represented in terms of inner product namely $f(x) = < x, z >$ where z depends on f and f has norm$\| z \| = \| f \|$.

Remark: For a proof of Theorem 1.5 refer to [2].

Theorem 1.6: In a Hilbert space a sequence x_n converges weakly $x_n \xrightarrow{w} x$ if and only if $< x_n, x > \to < x, z >$ for all z in the space as $n \to \infty$. Similarly, $< x_n - x, z > \to 0$ as $n \to \infty$ for all z in the space. This follows immediately from the Riesz's representation theorem.

Theorem 1.7(Pythagoras): If $(V, <.>)$ is an inner product space, $x, y \in V$ and $< x, y > = 0$ then $\| x + y \|^2 = \| x - y \|^2 = \| x \|^2 + \| y \|^2$.

Lemma 1.1: If A is an orthonormal set in an inner product space V, $f \in V$ and $a_1, a_2, \ldots, a_k$ are finitely many elements of A, then $\sum_{n=1}^{k} | < f, a_n > |^2 \leq \| f \|^2$.

Remark: let $x = \sum_{n=1}^{k} < f, a_n > a_n, \ y = f - x$.

Then $< x, y > \ = \ < \sum_{n=1}^{k} < f, a_n > a_n, \ f - \sum_{n=1}^{k} < f, a_n > a_n >$

$$= \sum_{n=1}^{k} < f, a_n >< a_n, f > - \sum_{n=1}^{k} < f, a_n > \overline{< f, a_n >}$$

$$= \sum_{n=1}^{k} | < f, a_n > |^2 - \sum_{n=1}^{k} | < f, a_n > |^2 = 0$$

Hence $\sum_{n=1}^{k} | < f, a_n > |^2 = \| x \| \leq \| x \|^2 + \| y \|^2 = \| f \|^2$ (by Pythagoras theorem).

Theorem 1.7(Bessel inequality): If $\{a_n | n = 1,2, \ldots\}$ is a denumerable set in an inner product space V and $f \in V$ then $\sum_{n=1}^{\infty} | < f, a_n > |^2 \leq \| f \|^2$.

Remark: The proof follows from the lemma above by letting $k \to \infty$.

Definition1.6 (Pointwise Convergence): Let S be a nonempty set. Suppose f_n is sequence of real-valued functions on X. Given $a \in S$ we say that $f_n(a)$ is convergent if there exists a real-valued function f such that these conditions are satisfied: to every positive real number ε there corresponds a positive integer t such that $|f_n(a) - f(a)| < \varepsilon$ for all $n \geq t$.

Definition 1.7 (Uniform Convergence): Let X be a nonempty set and f_n a sequence of functions on X. We say that f_n is uniformly convergent on X if there exists a function f on X satisfying the condition: to every positive real number ε there corresponds a positive integer p such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X$ and for all $n \geq p$.

Remark: According to the above definition we have to try to get p, independent of x, which is not easy in practice. This method can be replaced by an easy method given in the following theorem.

Theorem 1.8 [M_n -Test]: Let f_n be a sequence of functions defined on an interval S such that $\lim_{n\to\infty} f_n = f(x)$ for all $x \in [\alpha, \beta]$ and also let $M_n = sup\{|f_n(x) - f(x)|\}$ for all $x \in [\alpha, \beta]$. Then f_n converges uniformly on $[\alpha, \beta]$ if and only if $M_n \to 0$ as $n \to \infty$.

Example 1.1: Prove that the sequence $f_n = \frac{x}{(1+nx^2)}$ converges uniformly on any closed interval S.

Remark: the Pointwise limit $f(x) = \lim_{n\to\infty} f_n(x) = 0$, for all $x \in S$.

Therefore $|f_n(x) - f(x)| = |\frac{x}{(1+nx^2)} - 0| = |\frac{x}{(1+nx^2)}|$, if we let $y = \frac{x}{(1+nx^2)}$ then $\frac{dy}{dx} = \frac{1-nx^2}{(1+nx^2)^2}$. Now for maximum and minimum value of y, we have $\frac{dy}{dx} = 0$ which implies that $x = \frac{1}{\sqrt{n}} \in S$. $\frac{d^2y}{dx^2} = \frac{-2nx(1+nx^2)-4nx(1-nx^2)}{(1+nx^2)^3}$, substituting $x = \frac{1}{\sqrt{n}}$ we get $\frac{d^2y}{dx^2} = -\left(\frac{\sqrt{n}}{2}\right) < 0$. This shows that y has maximum when $x = \frac{1}{\sqrt{n}}$ with value $y = \frac{1}{2\sqrt{n}}$.

Therefore $M_n = sup\{|f_n(x) - f(x)|\} = sup\{|y|\} = \frac{1}{2\sqrt{n}}$ for all $\in [\alpha, \beta]$. Since $M_n \to 0$ as $n \to \infty$ the sequence $f_n = \frac{x}{(1+nx^2)}$ converges uniformly on any closed interval.

Definition 1.8 (Weak convergence): A sequence x_n in a normed space $(X, \|,\|)$ is said to be weakly convergent if there is an $x \in X$ such that for every $f \in X'$, the dual space of X, $\lim_{n\to\infty} f(x_n) = f(x)$ and it is denoted by $x_n \xrightarrow{w} x$ and x is called weak limit of the sequence x_n.

Remark: Sequence x_n is the points in Xso a vector quantity and f is a bounded linear functional defined on X. When we say x_n converges to x means the corresponding sequence of scalars. Scalars are obtained by taking the images of x_n under f. So $f(x_n)$ is a sequence of scalars. When such a sequence converges then we say it is weakly convergent.

Lemma 1.2: Let x_n be a weakly convergent sequence in a normed space $(X, \|,\|)$ and $x_n \xrightarrow{w} x$ then :

(a) the weak limit x of x_n is unique,

(b) every subsequence of x_n converges weakly to x.

Remark: Given $x_n \xrightarrow{w} x$ means $f(x_n) \to f(x)$ for all $f \in X'$.

Suppose that the weak limit is not unique, then there exist a y such that $x_n \xrightarrow{w} y$ i.e. $f(x_n) \to f(y)$ for all $f \in X'$.Then $f(x) - f(y) = f(x - y)$ since f is linear. That implies $f(x) - f(y) = 0$ because $f(x_n)$ is a sequence of scalars and such sequence has its limit being unique. Therefore $f(x) = f(y)$. Now that also implies$f(x - y) = 0$ for all $f \in X'$. Hence $x - y = 0$ by the corollary of the Hahn Banach Theorem.

(b). Given $x_n \xrightarrow{w} x$ then $f(x_n) \to f(x)$ for all $f \in X'$. But $f(x_n)$ is a convergent sequence of scalars and so all of its subsequences will converge to the same limit point.

Definition 1.9 (Strong convergence): A sequence x_n in a normed space $(X, \|,\|)$ is strongly convergent if there exists an $x \in X$ such that $\lim_{n\to\infty} \| x_n - x \| = 0$ or $x_n \xrightarrow{\|,\|} x$ as $n \to \infty$.

Section Two: uniform convergence with integration and differentiation

Theorem 2.1: Let f be a real-valued function on a closed interval$[a, b]$ and f_n a sequence of real-valued functions on $[a, b]$ such that $f_n \to f$ uniformly as $n \to \infty$. If f_n is integrable over $[a, b]$ in the Riemann sense for all $n \geq t$ then f is integrable over $[a, b]$ in the Riemann sense and $\int_a^b f_n(x) \to \int_a^b f(x)dx$ as $n \to \infty$.

Example 2.1: Euler number: Historically Euler's constant γ is defined as $\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - In\, n\right]$. We proceed to use uniform convergence to give in depth description of γ. $\frac{x}{k(k+x)} \leq \frac{1}{k^2}$ for all $x \in [0,1]$ and for every positive integer k. Also $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent. Hence $\sum_{k=1}^{\infty} \frac{x}{k(k+x)}$ is uniformly convergent on $[0,1]$ by Weierstrass test. It follows that $\int_0^1 \sum_{k=1}^{n} \frac{x}{k(k+x)} \to \int_0^1 \sum_{k=1}^{\infty} \frac{x}{k(k+x)}$ as $n \to \infty$. On the other hand

$\int_0^1 \sum_{k=1}^{n} \frac{x}{k(k+x)} dx = \int_0^1 \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+x}\right) dx = \sum_{k=1}^{n} \frac{1}{k} - In\,(n+1)$ for every positive integer n.

Hence $\sum_{k=1}^{n} \frac{1}{k} - In\,(n+1)$ is convergent and $\sum_{k=1}^{n} \frac{1}{k} - In\,(n+1) \to \int_0^1 \sum_{k=1}^{\infty} \frac{x}{k(k+x)} dx$ as $n \to \infty$. Finally $\lim_{n\to\infty} \left\{\sum_{k=1}^{n} \frac{1}{k} - In\, n\right\} = \lim_{n\to\infty} \left\{\sum_{k=1}^{n} \frac{1}{k} - In\,(n+1)\right\} = \int_0^1 \sum_{k=1}^{\infty} \frac{x}{k(k+x)} dx$ because $In\, \frac{n+1}{n} \to 0$ as $n \to \infty$.

Theorem2.2: Let G be an open set in the Euclidean line R such that $[a, b] \subset G$ and F_n a sequence of class C^1 real-valued functions on G. If F and f are real-valued functions on $[a, b]$ such that $F_n(x) \to F(x)$ as $n \to \infty$ at all points $x \in [a, b]$ and $F_n' \to f$ uniformly on $[a, b]$ as $n \to \infty$, then F is differentiable at all points of $]a, b[$ and the derivative $F'(x) = f(x)$ for all $x \in]a, b[$.

Example 2.2: Suppose that $0 < r \leq \infty$ and r is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. Then r is the radius of convergence of $\sum_{n=1}^{\infty} n a_n x^{n-1}$. If $F(x) = \sum_{n=0}^{\infty} a_n x^n$ for all $x \in]-r, r[$ then F is differentiable at all points of $]-r, r[$ and $F'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for all $x \in]-r, r[$.

Remark: Given$x \in]-r, r[$ choose real numbers a, b such that $a < b$, $[a, b] \subset]-r, r[$ and $x \in]a, b[$. Let $s_k(x) = \sum_{n=0}^{k} a_n x^n$ for all $x \in]-r, r[$ and every positive integer k. Then s_k is differentiable at all points of R and the derivative $s_k'(x) = \sum_{n=0}^{k} n a_n x^{n-1}$ for all $x \in R$. Furthermore $\sum_{n=0}^{k} a_n x^n \to \sum_{n=0}^{\infty} a_n x^n$ uniformly on $[a, b]$ and $\sum_{n=1}^{k} a_n x^n \to \sum_{n=1}^{\infty} n a_n x^{n-1}$ uniformly on $[a, b]$ as $k \to \infty$. Hence F is differentiable at all points of $]-r, r[$ and $F'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for all $x \in]-r, r[$.

Remark: In the lemma below we present an alternative description of the exponential function e^x, defined by the formula $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and the radius of convergence of the power series is ∞.

Lemma 2.1: If k is a positive integer and $u_1, \dots, u_k \in [0,1]$ then $1 - (1-u_1) \dots (1-u_k) \leq u_1 + \cdots + u_k$

Remark: We proof by mean value Theorem: If $k = 1$ then the result is $1 - (1 - u_1) = u_1$.

If $k \geq 2$ define $f(x_1, \dots, x_k) = x_1 + \cdots + x_k - \{1 - (1 - x_1), \dots, (1 - x_k)\}$ on Euclidean space R^k.

Let $D = \{(x_1, \dots, x_k) \in R^k | \, 0 \leq x_1 \leq 1, \dots, 0 \leq x_k \leq 1\}$.Then f of class C^1 on R^k and

$\frac{\partial f}{\partial x_j} = 1 - \frac{(1-x_1),\dots,(1-x_k)}{1-x_j} \geq 0$ for every $j \in \{1, \dots, k\}$ and all points $(x_1, \dots, x_k) \in D$. Also $f(0, \dots, 0) = 0$.

If $(u_1, \dots, u_k) \in D$ then by the mean value theorem there exists $\theta \in [0,1]$ such that

$f(u_1, \dots, u_k) = f(u_1, \dots, u_k) - f(0, \dots, 0) = \sum_{j=1}^{k} u_j \frac{\partial f}{\partial x_j} (\theta u_1, \dots, \theta u_k) \geq 0.$

Thus $u_1 + \cdots + u_k \geq 1 - (1 - u_1) \dots (1 - u_k)$.

Remark: The proof can also be done by induction which will require elementary calculations.

Theorem 2.3: If x is a positive real number then $\left(1 + \frac{x}{n}\right)^n \to e^x$ as $n \to \infty$.

Remark: By the binomial theorem:

$$\left(1 + \frac{x}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + x + \sum_{k=2}^{n} \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{k-1}{n}\right) \frac{x^k}{k!}$$

$$\leq 1 + x + \sum_{k=2}^{n+1} \left(1 - \frac{1}{n+1}\right) \dots \left(1 - \frac{k-1}{n+1}\right) \frac{x^k}{k!} \quad \text{for all } n \geq 2.$$

It follows that $\left(1+\frac{x}{n}\right)^n$ is a monotonic non-decreasing sequence and $\left(1+\frac{x}{n}\right)^n \leq e^x$ for every positive integer n. Thus $\left(1+\frac{x}{n}\right)^n$ is convergent.

Furthermore $\left(1+\frac{x}{n}\right)^n - \sum_{k=0}^{n}\frac{x^k}{k!} = \sum_{k=2}^{n}\left(1-\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)\right)\frac{x^k}{k!}$ for all $n \geq 2$.

For each fixed positive integer $n > k$ the preceding lemma shows that

$1-\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right) \leq \frac{1+\cdots+(k-1)}{n}$ and so $1-\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right) \to 0$ as $n \to \infty$.

The conclusion is that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x$

Remark: Indeed if δ is a positive real number then $\left(1+\frac{x}{n}\right)^n$ converges uniformly on $[-\delta,\delta]$ to e^x.

Section Three:

comparison

Pointwise and Uniform convergence: Every Pointwise convergent sequence is uniformly convergent.

Remark: The conditions of Pointwise convergence and uniform convergence are quite different although every sequence f_n that converges uniformly to a function f will certainly converge Pointwise to f. The Pointwise limit is the same as the uniform limit, so when we are asked to show uniform convergence we have to first show that it converges Pointwise and then go on to show uniform convergence. The only difference between them is that in the definition of Pointwise convergence we are concerned only with one value of x at a time, the N we choose is thus allowed to depend not only ε but also on the point x itself. In the definition of uniform convergence, there must exist a single N which makes $|f_n(x)-f(x)| < \varepsilon$ for all $x \in X$. Thus uniform convergence is a stronger condition than Pointwise convergence.

Example 3.1: The sequence $f_n(x) = x + x^n, for\ all\ x \in]0,1[$ then

(a) $f_n(x) \to x$ as $n \to \infty$ Pointwise at each point $x \in]0,1[$.

(b) We however note that f_n is not uniformly convergent on $]0,1[$.

Remark: Assume that f_n is uniformly convergent on $]0,1[$. Let $I(x) = x$ for all $x \in]0,1[$. Then $f_n \to I$uniformly on $]0,1[$ as $n \to \infty$. Choose a positive integer p such that $|f_n(x) - I(x)| < \frac{1}{2}$ for all $x \in]0,1[$ and all $n \geq p$. Let $w = \left(\frac{1}{2}\right)^{\frac{1}{p}}$. Then $w \in]0,1[$, and so there is a contradiction $\frac{1}{2} = |f_n(w) - I(w)| < \frac{1}{2}$. The assumption is false. Hence f_n is not uniformly convergent on $]0,1[$.

Finally if $0 < \tau < 1$ then f is uniformly convergent on $]0,1[$.

Remark: $\tau^n \to 0$ as $n \to \infty$. Given a positive real number ε choose a positive integer q such that $\tau^n < \varepsilon$ for all $n \geq q$. Then $|f_n(x) - I(x)| = x^n \leq \tau^n < \varepsilon$ for all $x \in]0, \tau]$ and all $n \geq q$.

Theorem 3.1: A strongly convergent sequence is weakly convergent but the reverse in not generally true.

Remark: Given that $x_n \xrightarrow{\|.\|} x$ i.e. $\lim_{n\to\infty} \| x_n - x \| = 0$ as $n \to \infty$.

Then $|f(x_n) - f(x)| = |f(x_n - x)|$ since f is linear

$\leq \| f \| \| x_n - x \|$ since f is bounded for every $f \in X'$

$\to 0$ as $n \to \infty$.

Therefore $x_n \xrightarrow{w} x$ and completes the first part of the theorem.

Let $X = H$, Hilbert space, e_n be an orthonormal sequence in a Hilbert space H. Let $f \in H'$ i.e. f is a bounded linear functional in H. By Riesz's representation theorem a bounded linear functional f can be represented by $f(x) = < x, z >$, z is uniquely determined by f with $\| f \| = \| z \|$. So $f(e_n) = < e_n, z >$.

Using Bessel's inequality we have $\sum_{n=1}^{\infty} | < e_n, z > |^2 \leq \| z \|^2$. $\| z \|^2$ is finite and so the series by corollary 1.1 is convergent. Thus $\sum_{n=1}^{\infty} | < e_n, z > |^2 < \infty$. Hence by corollary 1.2 we have $< e_n, z > \to 0$ as $n \to \infty$ for all $z \in H$. Also $< 0, z > = 0$ for all $z \in H$.

Hence $< e_n, z > \to < 0, z >$ i.e. $< e_n - 0, z > = 0$ Therefore e_n is weakly convergent.

The orthonormal sequence is not strongly convergent since $\| e_n \| = 1$ such that $\| e_n - e_m \|^2 = < e_n - e_m, e_n - e_m > = 2$ and $\| e_{n-} e_m \| = \sqrt{2}$

Also $\lim_{n,m\to\infty} \| e_{n-} e_m \| = \sqrt{2} \neq 0$ and so its not Cauchy and hence does not converge strongly. This completes the proof.

Example: Let$f \in L_2(0,2\pi)$. Then we know that the Fourier series of f converges in $L_2(0,2\pi)$. Therefore the Fourier coefficients converge to zero, and in particular $\int_0^{2\pi} f(x) \sin(nx) dx \to 0 \quad \forall f \in L_2(0,2\pi)$.

This result is known as the Riemann-Lebesgue lemma. Thus the sequence $\{\sin(nx) | n \geq 1\}$ converges weakly to 0 in$L_2(0,2\pi)$. But certainly the sequence does not converge strongly to 0in $L_2(0,2\pi)$.

Theorem 3.2: In finite dimensional space weakly convergent sequence implies strong convergence.

Remark: Let $dimX = n$ and $e_1, e_2, \ldots, e_n$ be the basis elements of X. Let $x_m, x \in X$ and so

$x_m = \alpha_1^m e_1 + \alpha_2^m e_2 + \cdots + \alpha_n^m e_n$ and $x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$.

Given $x_n \overset{\|.\|}{\to} x$ implies that $x_n \overset{w}{\to} x$ by the theorem proved above.

Let $x_n \overset{w}{\to} x$ i.e. $f(x_n) \to f(x)$ for all $f \in X'$. In particular $f_1, f_2, \dots, f_n \in X'$ where

$$f_i(e_k) = \delta_{ik} = \begin{cases} 1, i = k \\ 0, i \neq k \end{cases}$$

$f_1, f_2, \dots, f_n$ will form a dual basis. So $f_j(x_m) = \alpha_j^m$ and $f_j(x) = \alpha_j$.

Now $f_j(x_m) \to f_j(x) \implies \alpha_j^m \to \alpha_j$ as $m \to \infty$.

Therefore $\| x_{m-} x \| = \| \sum_{j=1}^{m} (\alpha_j^m - \alpha_j)\, e_j \|$

$\leq \sum_{j=1}^{m} |\alpha_j^m - \alpha_j| \, \| e_j \| \to 0$ as $m \to \infty$ i.e. the series approaches zero as $m \to \infty$

$\implies \| x_{m-} x \| = 0$ as $m \to \infty$. Thus $x_m \to x$ strongly.

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