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Estimation of Cauchy Parameters under Ranked Set Sampling

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Abstract

In this paper, we will consider the MLEs (maximum likelihood estimators), and meds (median estimators) of the parameters of the Cauchy distribution using SRS (simple random sample) and RSS (ranked set sample). A comparison between these estimators using bias, MSE (mean square error) and the efficiency will be done using simulation. It appears that the MLE based on RSS can be a real competitor to the MLE based on SRS.

Keywords: Ranked set sampling; Simple random sampling; parameters; Cauchy distribution; maximum likelihood estimator; bias; mean square error.

1. Introduction

The Cauchy distribution is one of the interesting continuous distributions. There are many areas of application of the Cauchy distribution including economic modeling (time series analysis, stock and commodity price changes, and sales), theory of atomic and nuclear transitions and the Brownian motion tend to a Cauchy distribution. The Cauchy distribution gains its importance by giving a good approximation to the normal distribution.

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The problem of estimation of the unknown parameters of the Cauchy distribution is considered by many authors under simple random sampling (SRS) and ranked set sample (RSS). References [3,5,8,10,13,14,15] considered some inferences for the Cauchy distribution based on maximum likelihood estimators (MLEs).

Reference [9] proposed a new estimation technique, called windows estimates and calculated the efficiencies for these estimates of the parameters of the Cauchy distribution. Reference [11] estimated the Location and Scale in Cauchy distributions using the Empirical Characteristic Function (ECF). Reference [18] considered the MLE for the Location Parameter of a Cauchy distribution. Reference [6] derived the Bayesian estimators of the Cauchy parameters. Reference [16] proposed blue estimator of the location parameter of a Cauchy distribution based on RSS.

RSS as introduced by Reference [4] is an ingenious sampling technique for selecting a sample which is more informative than a SRS to estimate the population mean. He used of RSS technique to estimate the mean pasture and forage yields. RSS technique is very useful when visual ranking of population units is less expensive than their actual quantifications. Therefore, selecting a sample based on RSS can reduce the cost and increase the efficiency of estimation.

The basic idea behind selecting a sample under RSS can be described as follows: Select *m* random samples each of size *m*. Using a visual inspection or any cheap method to rank the units within each sample with respect to the variable of interest. Then select, for actual measurement, the i^{th} smallest unit from the i^{th} sample, $i = 1,...,m$. In this way, we obtain a total of m measured units, one from each sample. The procedure could be repeated r times until a sample of $n = mr$ measurements are obtained. These mr measurements form RSS. Reference [12] gave the theoretical background for RSS. They showed that the mean of an RSS is an unbiased estimator of the population mean with variance smaller than that of the mean of a SRS. Reference [2] showed that the RSS mean remains unbiased and more efficient than the SRS mean for estimating the population even if ranking is not perfect. A comprehensive survey about developments in RSS can be found in References [17, 7].

In this paper, since there are many attractive applications of Cauchy distribution, it would be of interest to conduct some statistical inference for Cauchy distribution. The statistical inference includes the study of some properties of Cauchy distribution, emphasizing on estimation of Cauchy parameters. The estimation of the location and scale parameters denoted as α and β respectively, of the Cauchy distribution under SRS and RSS is studied.

The Cauchy parameters are estimated by using several methods of estimation in both cases of SRS and RSS such as maximum likelihood and median. Furthermore, the performance of these estimators is investigated and compared through simulation. Bias, mean square error (MSE) and efficiency of these estimators are used for comparison.

The Cauchy distribution is a special form of the Pearson Type VII distribution and it is a member of t-family which has 1 as the degrees of freedom. It is symmetric around the location parameter and looks like the normal except for the heavy tails. The cdf and pdf of the random variable X which has a Cauchy distribution with parameters α and β are given respectively by

$$F(x;\alpha,\beta)=\frac{1}{\pi}\tan^{-1}\left(\frac{x-\alpha}{\beta}\right)+\frac{1}{2}, \tag{1}$$

$$f(x;\alpha,\beta)=\frac{1}{\pi\beta\left(1+\left(\frac{x-\alpha}{\beta}\right)^2\right)}, \tag{2}$$

where α is the location parameter and β is the scale parameter, $\beta>0,\ x$ and $\alpha\in(-\infty,\infty)$, denoted as $X\sim C(\alpha,\beta)$. Let $X_1, X_2,...,X_n$ be a random sample from X. We will study the MLEs, meds (median estimators) and blues (best linear unbiased estimators) in case of one parameter is unknown and the other is known based on $X_1, X_2,...,X_n$.

2. Material and methods

Assume $X_1, X_2,...,X_n$ is a SRS from a Cauchy distribution. We will study several estimators of α when β is known based on $X_1, X_2,...,X_n$. Since our distribution does not have a mean, we will propose the following estimators of the location parameter α .

a) Estimators of α when β is known using SRS

1. MLE:

Let $X_1, X_2,...,X_n$ be a random sample from (2). The log-likelihood function is given by

$$l(\alpha,\beta)=-n\log(\pi)-n\log(\beta)-\sum_{i=1}^{n}\log\left(1+\left(\frac{X_i-\alpha}{\beta}\right)^2\right). \tag{3}$$

After taking the derivative with respect to α and equating to 0, we obtain the MLE as:

$$\sum_{i=1}^{n}\frac{\left(\frac{X_i-\hat{\alpha}_{MLE,S}}{\beta}\right)}{\left(1+\left(\frac{X_i-\hat{\alpha}_{MLE,S}}{\beta}\right)^2\right)}=0. \tag{4}$$

2.2 *med:*

Let m_α be the population median of X, then

$$F(m_\alpha) = \frac{1}{\pi}\tan^{-1}\left(\frac{m_\alpha - \alpha}{\beta}\right) + \frac{1}{2} = \frac{1}{2}. \quad (5)$$

Then replace m_α by the median in SRS, i.e. $x_{med,S}$, we get

$$\hat{\alpha}_{med,S} = X_{med,S}. \quad (6)$$

b) Estimation of β when α is known using SRS

Let $X_1, X_2, \ldots, X_n$ is a SRS from. We will study several estimators of β when α is known based on $X_1, X_2, \ldots, X_n$.

1. MLE:

Let $X_1, X_2, \ldots, X_n$ be a random sample from (2). The log-likelihood function is given by

$$l(\alpha, \beta) = -n\log(\pi\beta) - \sum_{i=1}^{n}\log\left(1 + \left(\frac{X_i - \alpha}{\beta}\right)^2\right). \quad (7)$$

After taking the derivative with respect to β and equating to 0, we obtain the MLE as:

$$\sum_{i=1}^{n}\frac{\left(\frac{X_i - \alpha}{\beta_{MLE,S}}\right)^2}{\left(1 + \left(\frac{X_i - \alpha}{\beta_{MLE,S}}\right)^2\right)} = \frac{n}{2}. \quad (8)$$

***2. med*:**

Let m_α be the population median of X, then

$$F(m_\alpha) = \frac{1}{\pi}\tan^{-1}\left(\frac{m_\alpha - \alpha}{\beta}\right) + \frac{1}{2} = \frac{1}{2}. \quad (9)$$

Then replace m_α by the median in SRS, i.e. $x_{med,S}$, we get

$$\hat{\beta}_{med,S} = X_{med,S}. \tag{10}$$

c) *Estimations of α when β is known using RSS*

Let $X_{(i:m)j}, i = 1,...m$ and $j = 1,...,r$ denote the i^{th} order statistics from the i^{th} set of size *m* of the j^{th} cycle be the RSS data for X with sample size $n = mr$. We will study several estimators of α when β is known. Since our distribution does not have a mean, we will propose the following estimators of the location parameter α .

1. *MLE:*

Using (1.1) and (1.2), the pdf of $X_{(i:m)j}$ is given by reference [1]

$$f_{i:m}(X_{(i:m)j}) = c\left(F(X_{(i:m)j})\right)^{i-1}\left(1-F(X_{(i:m)j})\right)^{m-i} f(X_{(i:m)j}), \tag{11}$$

where $c = \dfrac{1}{B(i, m-i+1)}$, $f(X_{(i:m)j}) = \dfrac{1}{\pi\beta\left(1+\left(\dfrac{X_{(i:m)j}-\alpha}{\beta}\right)^2\right)}$ and

$$F(X_{(i:m)j}) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{X_{(i:m)j}-\alpha}{\beta}\right).$$

Then the likelihood function is given by

$$l(\alpha,\beta) = \prod_{i=1}^{r}\prod_{j=1}^{m} f_{i:m}(X_{(i:m)j})$$

$$= \prod_{i=1}^{r}\prod_{j=1}^{m}\left[c\left(\frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{X_{(i:m)j}-\alpha}{\beta}\right)\right)^{i-1}\left(\frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{X_{(i:m)j}-\alpha}{\beta}\right)\right)^{m-i}\left(\pi\beta\left(1+\left(\frac{X_{(i:m)j}-\alpha}{\beta}\right)^2\right)\right)^{-1}\right]$$

(12)

After using the log-likelihood function of (12), take the derivative of (12) with respect to α and equating the resulting quantity to zero. Since there is no explicit solution for (12), the equation needs to be solved numerically to find $\hat{\alpha}_{MLE,R}$.

3. *Adhoc Estimator*:

These are the same as the estimators in (5) and (6) with SRS is replaced by RSS.

4. *med*:

The median in RSS is given by

$$\hat{\alpha}_{med,R} = X_{(med:m)j}. \tag{13}$$

d) Estimation of β when α is known using RSS

Let $X_{(i:m)j}, i = 1,...m$ and $j = 1,...,r$ denote the i^{th} order statistics from the i^{th} set of size m of the j^{th} cycle be the RSS data for X with sample size $n = mr$. We will study several estimators of β when α is known. Since our distribution does not have a mean, we will propose the following estimators of the scale parameter β.

1. MLE:

After using the log-likelihood function of (12), take the derivative of (12) with respect to β and equating the resulting quantity to zero. Since there is no explicit solution for (12), the equation needs to be solved numerically to find $\hat{\beta}_{MLE,R}$.

2. med:

The median in RSS is given by

$$\hat{\beta}_{med,R} = X_{(med:m)j}. \tag{14}$$

5. Results

In this section, a comparison between all above estimators for both parameters of the Cauchy distribution is carried out under SRS and RSS using simulation. The package R has been used to conduct the simulation. The following values of the parameters and sample sizes have been considered:

$\alpha = 0.5,\ \beta = 1; \alpha = 1,\ \beta = 0.5; \alpha = 1,\ \beta = 1; \alpha = 1,\ \beta = 2; \alpha = 2,\ \beta = 1,\ n = 12$ and $n = 24$.

For each n, a set $(m;r)$ is decided such that $n = mr$. The bias and the MSE are computed for all the estimators under consideration. The efficiency between all estimators with respect to the MLE based on SRS are calculated where the efficiency of the estimator is defined as:

$$eff\ (\hat{\theta}_2, \hat{\theta}_1) = \frac{MSE\,(\hat{\theta}_1)}{MSE\,(\hat{\theta}_2)} \text{ where } MSE\,(\hat{\theta}_2) = \frac{1}{10,000}\sum_{t=1}^{10,000}\left(\hat{\theta}_{2t} - \theta_2\right)^2.$$

If $eff\ (\hat{\theta}_2, \hat{\theta}_1) > 1$ then $\hat{\theta}_2$ is better than $\hat{\theta}_1$.

Table 1: The biases and MSE of the estimators with respect $\hat{\alpha}_{mle,S}$

			Bias				MSE			
$(\alpha,\beta$	n	n=mr	$\hat{\alpha}_{mle,S}$	$\hat{\alpha}_{med,S}$	$\hat{\alpha}_{med,adhoc}$	$\hat{\alpha}_{mle,R}$	$\hat{\alpha}_{mle,S}$	$\hat{\alpha}_{med,S}$	$\hat{\alpha}_{med,adhoc}$	$\hat{\alpha}_{mle,R}$
(1,1)	12	m=2, r=6	0.005	0.021	0.009	0.011	0.216	0.256	0.185	0.155
		m=3, r=4			0.002	0.000			0.134	0.115
		m=4, r=3			0.004	0.007			0.111	0.098
	24	m=2, r=12	0.002	0.003	0.001	0.006	0.096	0.104	0.081	0.067
		m=3, r=8			0.006	0.001			0.066	0.054
		m=4, r=6			0.001	0.001			0.056	0.044
(1,2)	12	m=2, r=6	0.003	0.004	0.004	0.025	0.850	1.093	0.704	0.619
		m=3, r=4			0.007	0.005			0.561	0.470
		m=4, r=3			0.006	0.018			0.459	0.403
	24	m=2, r=12	0.002	0.000	0.009	0.005	0.366	0.441	0.328	0.284
		m=3, r=8			0.002	0.006			0.270	0.224
		m=4, r=6			0.004	0.003			0.225	0.158
(2,1)	12	m=2, r=6	0.031	0.002	0.004	0.009	0.225	0.267	0.183	0.091
		m=3, r=4			0.011	0.012			0.137	0.108
		m=4, r=3			0000	0.005			0.118	0.102
	24	m=2, r=12	0.002	0.003	0.007	0.005	0.097	0.111	0.079	0.067
		m=3, r=8			0.002	0.002			0.066	0.055
		m=4, r=6			0.002	0.001			0.057	0.043
(0.5, 1)	12	m=2, r=6	0.010	0.001	0.005	0.003	0.231	0.226	0.178	0.154
		m=3, r=4			0.002	0.008			0.139	0.125
		m=4, r=3			0.001	0.01			0.113	0.095
	24	m=2, r=12	0.010	0.003	0.008	0.001	0.097	0.117	0.079	0.070
		m=3, r=8			0.002	-0.006			0.066	0.054
		m=4, r=6			0.006	-0.002			0.054	0.045
(1,0. 5)	12	m=2, r=6	-0.003	0.005	0.003	0.004	0.057	0.066	0.043	0.040
		m=3, r=4			0.001	-0.014			0.034	0.029
		m=4, r=3			0.004	0.007			0.029	0.023
	24	m=2, r=12	-0.003	0.001	0.002	0.005	0.022	0.028	0.019	0.017
		m=3, r=8			0.001	0.003			0.017	0.013
		m=4, r=6			0.001	0.002			0.014	0.011

Table 2: The efficiencies of the estimators with respect to $\hat{\alpha}_{mle,S}$

(α,β)	n	$n=mr$	$\hat{\alpha}_{mle,S}$	$\hat{\alpha}_{med,S}$	$\hat{\alpha}_{med,adhoc}$	$\hat{\alpha}_{mle,R}$
(1,1)	12	m=2, r=6		0.84375	1.168	1.394
		m=3, r=4			1.612	1.878
		m=4, r=3	1		1.946	2.204
	24	m=2, r=12		0.923077	1.185	1.433
		m=3, r=8			1.455	1.778
		m=4, r=6	1		1.714	2.182
(1,2)	12	m=2, r=6		0.777676	1.207	1.373
		m=3, r=4			1.515	1.809
		m=4, r=3	1		1.852	2.109
	24	m=2, r=12		0.829932	1.116	1.289
		m=3, r=8			1.356	1.634
		m=4, r=6	1		1.627	2.316
(2,1)	12	m=2, r=6		0.842697	1.23	2.473
		m=3, r=4			1.642	2.083
		m=4, r=3	1		1.907	2.206
	24	m=2, r=12		0.873874	1.228	1.448
		m=3, r=8			1.47	1.764
		m=4, r=6	1		1.702	2.256
(0.5,1)	12	m=2, r=6		1.022124	1.298	1.500
		m=3, r=4			1.662	1.848
		m=4, r=3	1		2.044	2.432
	24	m=2, r=12		0.82906	1.228	1.386
		m=3, r=8			1.47	1.796
		m=4, r=6	1		1.796	2.156
(1,0.5)	12	m=2, r=6		0.863636	1.326	1.425
		m=3, r=4			1.676	1.966
		m=4, r=3	1		1.966	2.478
	24	m=2, r=12		0.785714	1.158	1.294
		m=3, r=8			1.294	1.692
		m=4, r=6	1		1.571	2

Table 3: The biases and MSE of the estimators with respect $\hat{\beta}_{mle,S}$

			Bias				MSE			
$(\alpha,\beta$	*n*	*n=mr*	$\hat{\beta}_{mle,S}$	$\hat{\beta}_{med,S}$	$\hat{\beta}_{med,adhoc}$	$\hat{\beta}_{mle,R}$	$\hat{\beta}_{mle,S}$	$\hat{\beta}_{med,S}$	$\hat{\beta}_{med,adhoc}$	$\hat{\beta}_{mle,R}$
(1,1)	12	m=2, r=6	0.083	0.001	0.004	0.077	0.242	0.258	0.172	0.204
		m=3, r=4			0.001	0.044			0.140	0.146
		m=4, r=3			0.005	0.053			0.112	0.138
	24	m=2, r=12	0.039	0.001	0.010	0.045	0.093	0.115	0.079	0.084
		m=3, r=8			0.064	0.022			0.064	0.070
		m=4, r=6			0.057	0.026			0.057	0.063
(1,2)	12	m=2, r=6	0.199	- 1.003	-1.009	0.102	1.081	2.071	1.755	0.687
		m=3, r=4			-0.994	0.073			1.534	0.582
		m=4, r=3			-1.005	0.088			1.481	0.536
	24	m=2, r=12	0.105	- 0.999	-1.003	0.097	0.429	1.442	1.330	0.380
		m=3, r=8			-0.999	0.100			1.273	0.331
		m=4, r=6			-1.007	0.022			1.244	0.242
(2,1)	12	m=2, r=6	0.101	1.002	0.993	0.052	0.258	1.269	1.161	0.170
		m=3, r=4			1.002	0.059			1.141	0.164
		m=4, r=3			0.998	0.046			1.111	0.139
	24	m=2, r=12	0.053	0.995	0.994	0.043	0.103	1.102	1.066	0.086
		m=3, r=8			1.008	0.032			1.081	0.072
		m=4, r=6			1.001	0.028			1.058	0.058
(0.5, 1)	12	m=2, r=6	0.110	- 0.496	-0.499	0.068	0.285	0.508	0.420	0.203
		m=3, r=4			-0.503	0.044			0.388	0.157
		m=4, r=3			-0.503	0.047			0.368	0.149
	24	m=2, r=12	0.046	- 0.499	-0.501	0.060	0.110	0.363	0.332	0.089
		m=3, r=8			-0.502	0.022			0.316	0.070
		m=4, r=6			-0.497	0.015			0.301	0.061
(1,0.5)	12	m=2, r=6	0.043	0.503	0.502	0.036	0.057	0.319	0.295	0.049
		m=3, r=4			0.497	0.032			0.280	0.039
		m=4, r=3			0.496	0.021			0.273	0.031
	24	m=2, r=12	0.020	0.503	0.504	0.009	0.023	0.281	0.274	0.018
		m=3, r=8			0.499	0.012			0.266	0.018
		m=4, r=6			0.499	0.012			0.263	0.017

Table 4: The efficiencies of the estimators with respect to $\hat{\beta}_{mle,S}$

(α,β)	*n*	*n=mr*	$\hat{\beta}_{mle,S}$	$\hat{\beta}_{med,S}$	$\hat{\beta}_{med,adhoc}$	$\hat{\beta}_{mle,R}$
(1,1)	12	m=2, r=6		0.938	0.0141	1.1863
		m=3, r=4			0.0173	1.6575
		m=4, r=3	1		0.0216	1.7536
	24	m=2, r=12		0.8087	0.0118	1.1071
		m=3, r=8			0.0145	1.3286
		m=4, r=6	1		0.0163	1.4762
(1,2)	12	m=2, r=6		0.522	0.0062	1.5735
		m=3, r=4			0.0070	1.8574
		m=4, r=3	1		0.0073	2.0168
	24	m=2, r=12		0.2975	0.0032	1.1289
		m=3, r=8			0.0034	1.2961
		m=4, r=6	1		0.0034	1.7727
(2,1)	12	m=2, r=6		0.2033	0.0022	1.5176
		m=3, r=4			0.0023	1.5732
		m=4, r=3	1		0.0023	1.8561
	24	m=2, r=12		0.0935	0.0010	1.1977
		m=3, r=8			0.0010	1.4306
		m=4, r=6	1		0.0010	1.7759
(0.5,1)	12	m=2, r=6		0.561	0.0068	1.4039
		m=3, r=4			0.0073	1.8153
		m=4, r=3	1		0.0077	1.9128
	24	m=2, r=12		0.303	0.0033	1.2360
		m=3, r=8			0.0035	1.5714
		m=4, r=6	1		0.0037	1.8033
(1,0.5)	12	m=2, r=6		0.1787	0.0019	1.1633
		m=3, r=4			0.0020	1.4615
		m=4, r=3	1		0.0021	1.8387
	24	m=2, r=12		0.0819	0.0008	1.2778
		m=3, r=8			0.0009	1.2778
		m=4, r=6	1		0.0009	1.3529

The bias and MSE of the estimators will be reported in Tables 1 and 3 and the efficiencies of the estimators will be reported in Tables 2 and 4.

From Tables 1 to 4, the conclusions derived based on the results can be summarized as follows:

- In general, the bias is small for all estimators. Therefore, all the estimators are considered as unbiased estimators for α.
- From Table 1, it can be noticed that the Med under SRS has the smallest bias as compared to the other estimators considered in most cases. In general, for all estimators of α under RSS, the bias is less than the case under SRS.
- For fixed α, the MSE of $\hat{\alpha}$ decreases as the sample size increases.
- It noticed that from Table 2 that MLE under RSS is the most efficient than the MLE based on SRS.
- The efficiency of the other estimator ($\hat{\alpha}_{med,S}$) is less one.
- In general, the bias is small for all estimators except for $\hat{\beta}_{med,adhoc}$.
- From Table 3, it can be noticed that the MLE under RSS has the smallest bias as compared to the other estimators. In general, for all estimators of β under RSS, the bias is less than the case under SRS.
- For fixed β, the MSE of $\hat{\alpha}$ decreases as the sample size increases.
- It noticed that from Table 4 that MLE under RSS is the most efficient than the MLE based on SRS.
- The efficiency of the other estimators ($\hat{\beta}_{med,adhoc}$ *and* $\hat{\beta}_{med,S}$) is less one.

6. Conclusions and recommendations

We conclude that most of estimators are unbiased. Since the MLEs under RSS are more efficient than the MLE under SRS, RSS is recommended in case ordering can be done visually or by a cheap method. The other estimators are not recommended.

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