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On a Generalized $\mathcal{B}$K – Recurrent Finsler Space

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Abstract

In the present paper, we introduced a Finsler space F_n whose Cartan's fourth curvature tensor K^i_{jkh} satisfies the condition $\mathcal{B}_m K^i_{jkh} = \lambda_m K^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh})$, $K^i_{jkh} \neq 0$, where λ_m and μ_m are non-zero covariant vectors field called *recurrence vector*. The space satisfying this condition will be called *a generalized* $\mathcal{B}K$-*recurrent space*. The purpose of this paper is to obtain Berwald covariant derivative of first order for the h(v)-torsion tensor H^i_{kh} and the deviation tensor H^i_h, also to show that K- Ricci tensor K_{jk}, the curvature vectors K_j, H_j and the curvature scalar H are non-vanishing in our space. We have shown some tensors behave as recurrent and we have obtained various identities in such space.

Keywords: Finsler space ; generalized $\mathcal{B}$K-recurrent space ; K- Ricci tensor.

1. Introduction

R. Verma [12] discussed recurrence property of Cartan's third curvature tensor R^i_{jkh}, S. Dikshit [10] discussed birecurrent of Berwald curvature tensor H^i_{jkh}, F. Y. A. Qasem [2] introduced and studied the recurrence condition of the curvature tensor U^i_{jkh} in the sense of Berwald.

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C. K. Mishra and G. Lodli [1] studied C^h – recurrent and C^v – recurrent Finsler space of second order. N. S. H. Hussien [9] introduced and discussed K^h –recurrent Finsler space, M. A. A. Ali [6], F. Y. A. Qasem and M. A. A. Ali [3] and M. A. H. Alqufial, F. Y. A. Qasem and M. A. A. Ali [7] studied K^h –birecurrent Finsler space. F. Y. A. Qasem and A. M. A. Hanballa [4] studied K^h – generalized birecurrent Finsler space. P.N. Pandey, S. Saxena and A. Goswani [11] introduced and studied generalized H- recurrent space in the sense of Berwald.

Let us consider an n-dimensional Finsler space F_n equipped with the metric function F(x,y) satisfies the request condition [5].

The relation between the metric function F and the corresponding metric tensor given by

$$(1.1)\qquad a)\ g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x,y) \qquad \text{and} \qquad b)\ g_{ij}(x,y)\ y_i y^i = F^2.$$

The tensor $g_{ij}(x,y)$ is symmetric and positively homogeneous of degree zero in y^i.

The vector y_i and its associative y^i satisfy the following relations

$$(1.2)\qquad a)\ g_{ij}(x,y)y^i = y_j \qquad \text{and} \qquad b)\ y_i y^i = F^2.$$

The two sets of quantities g_{ij} and its associative g^{ij}, which are components of a metric tensor are connected by

$$(1.3)\qquad a)\ g_{ij}g^{ik} = \delta_j^k = \begin{cases} 1 & if\ i = k, \\ 0 & if\ i \neq k \end{cases} \qquad \text{and} \qquad b)\ \delta_h^i g_{ik} = g_{hk}.$$

By differentiating (1.1a) partially with respect to y^k, we construct a new tensor C_{ijk} is defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_i g_{jk}.$$

This new tensor C_{ijk} is positively homogeneous of degree -1 in y^i and symmetric in all its indices and called *(h)hv-torsion tensor*[8]. A according to Euler's theorem on homogeneous functions, this tensor satisfies the following:

$$(1.4)\qquad C_{ijk}y^i = C_{jki}y^i = C_{kij}y^i = 0.$$

Berwald's covariant derivative of the vector y^i vanish identically, i.e.

$$(1.5)\qquad \mathcal{B}_k y^i = 0.$$

But, in general, Berwald's covariant derivative of the metric tensor g_{ij} does not vanish and given by

$$(1.6) \qquad \mathcal{B}_k g_{ij} = -2C_{ijk|h} y^h = -2y^h \mathcal{B}_h C_{ijk}.$$

The tensor K^i_{jkh} is called *Cartan's fourth curvature tensor*, it is positively homogeneous of degree zero in y^i, which defined by

$$K^i_{jkh} := \partial_h \Gamma^{*i}_{kj} + (\dot{\partial}_s \Gamma^{*i}_{jh}) G^s_k + \Gamma^{*i}_{th} \Gamma^{*t}_{kj} - h/k .^*$$

The curvature tensor K^i_{jkh} is skew-symmetric in its last two lower indices, i.e.

$$(1.7) \qquad K^i_{jkh} = -K^i_{jhk}.$$

The associate curvature tensor K_{ijkh} of the curvature tensor K^i_{jkh} is given by

$$(1.8) \qquad K_{ijkh} := g_{rj} K^r_{ikh}.$$

The curvature tensor K^i_{jkh} and its associative K_{ijkh} satisfy the following relations

$$(1.9) \qquad a) \; K_{ijkh} + K_{ijhk} = -2C_{ijs} H^s_{hk},$$

$$b) \; K_{jikh} + K_{jkih} + K_{jhki} + 2y^r \left(C_{jis} K^s_{rhk} + C_{jks} K^s_{rih} + C_{jhs} K^s_{rki} \right) = 0$$

and

$$c) \; K^i_{jki} = K_{jk}.$$

The curvature tensor K^i_{jkh} and the h(v)-torsion tensor H^i_{kh} are related by

$$(1.10) \qquad K^i_{jkh} y^j = H^i_{kh}.$$

The h(v)-torsion tensor H^i_{kh}, the deviation tensor H^i_h are positively homogeneous of degree one and two in y^i, respectively. In view of Euler's theorem on homogeneous functions and since the contraction of indies doesn't effect on the degree of the homogeneous, we have the following relations

$$(1.11) \qquad a) \; H^i_{kh} y^k = H^i_h = -H^i_{hk} y^k, \qquad b) \; H^i_{ki} = H_k \qquad \text{and} \qquad c) \; H^i_i = (n-1)H.$$

The tensor $H_{.rkh}$ defined by

$$(1.12) \qquad H_{.rkh} = K_{jrkh} y^j.$$

2. A Generalized $\mathcal{B}K$ -Recurrent Space

Let us consider a Finsler space F_n in which Cartan's fourth curvature tensor K^i_{jkh} satisfies the generalized recurrence property with respect to Berwald's connection parameters G^i_{kh}, i.e. characterized by the following condition

$$\mathcal{B}_m K^i_{jkh} = \lambda_m K^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right), \qquad K^i_{jkh} \neq 0, \tag{2.1}$$

* $-h/k$ means the subtraction from the former term by interchanging the indices h and k .

where $\mathcal{B}_m$ is Berwald's covariant differential operator with respect to x^m, λ_m and μ_m are called *recurrence vectors*.

Definition 2.1. A Finsler space F_n in which Cartan's fourth curvature tensor K^i_{jkh} satisfies the condition (2.1), where λ_m and μ_m are non-zero covariant vectors field. Such space and tensor will be called *generalized $\mathcal{B}K$-recurrent space* and *generalized recurrent tensor*, respectively. We shall denote them briefly by $G\mathcal{B}K - RF_n$ and $G\mathcal{B}K - R$, respectively.

Let us consider an $G\mathcal{B}K - RF_n$ which characterized by the condition (2.1).

Transvecting the condition (2.1) by y^j, using (1.10), (1.5a) and (1.2a), we get

$$\mathcal{B}_m H^i_{kh} = \lambda_m H^i_{kh} + \mu_m\left(\delta^i_h y_k - \delta^i_k y_h\right). \tag{2.2}$$

Transvecting the condition (2.2) by y^k, using (1.11a), (1.5a) and (1.2b), we get

$$\mathcal{B}_m H^i_h = \lambda_m H^i_h + \mu_m\left(\delta^i_h F^2 - y^i y_h\right). \tag{2.3}$$

Thus, we conclude

Theorem 2.1. *In $G\mathcal{B}K - RF_n$, Berwald covariant derivative of first order for the h(v) - torsion tensor H^i_{kh} and the deviation tensor H^i_h are given by the conditions (2.2) and (2.3), respectively.*

Contracting the indices i and h in the condition (2.1) and using (1.9c), we get

$$\mathcal{B}_m K_{jk} = \lambda_m K_{jk} + (n-1)\mu_m g_{jk}. \tag{2.4}$$

Transvecting the condition (2.4) by y^j, using (1.5a) and (1.2a), we get

$$\mathcal{B}_m (K_{jk} y^j) = \lambda_m (K_{jk} y^j) + (n-1)\mu_m y_k. \tag{2.5}$$

Transvecting the condition (2.4) by y^k, using (1.5a), (1.2a) and putting ($K_{jk}y^k = K_j$), we get

$$(2.6) \qquad \mathcal{B}_m K_j = \lambda_m K_j + (n-1)\mu_m y_j.$$

Transvecting the condition (2.6) by y^j, using (1.2b) and (1.5a), we get

$$(2.7) \qquad \mathcal{B}_m (K_j y^j) = \lambda_m (K_j y^j) + (n-1)\mu_m F^2.$$

The conditions (2.4), (2.5), (2.6) and (2.7) show that K-Ricci tensor K_{jk}, the tensor $K_{jk}y^j$, the curvature vector K_j and the tensor $K_j y^j$ cannot vanish, because the vanishing of any one of them would imply $\mu_m = 0$, a contradiction.

Thus, we conclude

Theorem 2.2. *In $G\mathcal{B}K - RF_n$, K-Ricci tensor K_{jk} and the curvature vector K_j are non – vanishing.*

Contracting the indices i and h in the condition (2.2) and using (1.11b), we get

$$(2.8) \qquad \mathcal{B}_m H_k = \lambda_m H_k + (n-1)\mu_m y_k.$$

Contracting the indices i and h in the condition (2.3), using (1.11c) and (1.2b), we get

$$(2.9) \qquad \mathcal{B}_m H = \lambda_m H + \mu_m F^2.$$

The conditions (2.8) and (2.9) show that the curvature vector H_k and the curvature scalar H cannot vanish, because the vanishing of any one of them would imply $\mu_m = 0$, a contradiction.

Thus, we conclude

Theorem 2.3. *In $G\mathcal{B}K - RF_n$, the curvature vector H_k and the curvature scalar H are non – vanishing.*

Contracting the indices i and j in the condition (2.1) and using (1.3b), we get

$$\mathcal{B}_m K^s_{skh} = \lambda_m K^s_{skh}.$$

Thus, we conclude

Theorem 2.4. *In $G\mathcal{B}K - RF_n$, the tensor K^s_{skh} behaves as recurrent .*

Transvecting the condition (2.1) by g_{ir} , using (1.8) and (1.6), we get

(2.10) $$\mathcal{B}_m K_{jrkh} = \lambda_m K_{jrkh} + \mu_m (g_{jk} g_{hr} - g_{jh} g_{kr}) - 2K^i_{jkh} (y^t \mathcal{B}_t C_{irm}).$$

Thus, we conclude

Theorem 2.5. *In $G\mathcal{B}K - RF_n$, Berwald covariant derivative of first order for the associate curvature K_{ijkh} is given by the condition (2.10).*

Taking the covariant derivative for(1.9a) with respect to x^m *in the sense of Berwald*, we get

$$\mathcal{B}_m (K_{ijkh} + K_{ijhk}) = \mathcal{B}_m (-2C_{ijs} H^s_{hk}).$$

In view of (2.10) and by using (1.7), the above equation can be written as

$$\lambda_m (K_{ijkh} + K_{ijhk}) = \mathcal{B}_m (-2C_{ijs} H^s_{hk}).$$

By using (1.9a), the above equation becomes

$$\mathcal{B}_m (C_{ijs} H^s_{hk}) = \lambda_m (C_{ijs} H^s_{hk}).$$

Thus, we conclude

Theorem 2.6. *In $G\mathcal{B}K - RF_n$, the tensor $(C_{ijs} H^s_{hk})$ behaves as recurrent.*

Transvecting (2.10) by y^j and y^r, successively, using (1.12), (1.5a), (1.10) and (1.4), we get

$$\mathcal{B}_m (H_{.rkh} y^r) = \lambda_m (H_{.rkh} y^r).$$

Thus, we conclude

Theorem 2.7. *In $G\mathcal{B}K - RF_n$, the tensor $(H_{.rkh} y^r)$ behaves as recurrent.*

We shall obtain some identities which are satisfying in $G\mathcal{B}K - RF_n$, we know that, the associate curvature tensor K_{ijkh} satisfies ([1,2]) the identity (1.9b).

Using (1.10) in (1.9b), we get

(2.11) $$K_{jikh} + K_{jkih} + K_{jhki} + 2(C_{jis} H^s_{hk} + C_{jks} H^s_{ih} + C_{jhs} H^s_{ki}) = 0.$$

Taking the covariant derivative for (2.11) with respect to x^m in the sense of Berwald, we get

(2.12) $$\mathcal{B}_m (K_{jikh} + K_{jkih} + K_{jhki}) + 2(C_{jis} \mathcal{B}_m H^s_{hk} + C_{jks} \mathcal{B}_m H^s_{ih} + C_{jhs} \mathcal{B}_m H^s_{ki})$$

$$+2[(\mathcal{B}_m C_{jis})H^s_{hk} + (\mathcal{B}_m C_{jks})H^s_{ih} + (\mathcal{B}_m C_{jhs})H^s_{ki}] = 0.$$

Using the conditions (2.2), (2.10), the symmetric property of the metric tensor g_{ij}, $(C_{jis}\delta^s_k = C_{jik})$ and the symmetric property of the (h)hv-torsion tensor C_{jis} (in all its indices) in (2.12),we get

$$\text{(2.13)} \qquad \lambda_m[K_{jikh} + K_{jkih} + K_{jhki} + 2(C_{jis}H^s_{hk} + C_{jks}H^s_{ih} + C_{jhs}H^s_{ki})]$$

$$+2\mu_m(g_{ik}g_{hi} - g_{jh}g_{ki}) - 2y^t(K^p_{jkh}\mathcal{B}_t C_{pim} + K^p_{jih}\mathcal{B}_t C_{pkm}$$

$$+K^p_{jki}\mathcal{B}_t C_{phm}) + 2[(\mathcal{B}_m C_{jis})H^s_{hk} + (\mathcal{B}_m C_{jks})H^s_{ih} + (\mathcal{B}_m C_{jhs})H^s_{ki}] = 0.$$

Using (1.9b) in (2.13), we get

$$\text{(2.14)} \qquad \mu_m(g_{ik}g_{hi} - g_{jh}g_{ki}) - y^t(K^p_{jkh}\mathcal{B}_t C_{pim} + K^p_{jih}\mathcal{B}_t C_{pkm} + K^p_{jki}\mathcal{B}_t C_{phm})$$

$$+(\mathcal{B}_m C_{jis})H^s_{hk} + (\mathcal{B}_m C_{jks})H^s_{ih} + (\mathcal{B}_m C_{jhs})H^s_{ki} = 0.$$

Transvecting (2.14) by y^j, using (1.5a), (1.2a), (1.10) and (1.4), we get

$$\text{(2.15)} \qquad \mu_m(y_k g_{hi} - y_h g_{ki}) - y^t(H^p_{kh}\mathcal{B}_t C_{pim} + H^p_{ih}\mathcal{B}_t C_{pkm} + H^p_{ki}\mathcal{B}_t C_{phm}) = 0.$$

Transvecting (2.15) by y^i, using (1.5a), (1.2a), (1.4) and (1.11a), we get

$$\text{(2.16)} \qquad y^t(H^p_h\mathcal{B}_t C_{pkm} - H^p_k\mathcal{B}_t C_{phm}) = 0.$$

Thus, we conclude

Theorem 2.8. *In $G\mathcal{B}K - RF_n$, we have the identities (2.14), (2.15) and (2.16).*

In view of the condition (2.5) and (2.8), we get

$$\text{(2.17)} \qquad H_k = K_{jk}y^j.$$

In view of the condition (2.7) and (2.9), we get

$$\text{(2.18)} \qquad (n-1)H = K_j y^j.$$

Thus, we conclude

Theorem 2.9. *In $G\mathcal{B}K - RF_n$, we have the identities (2.17) and (2.18).*

In view of the conditions (2.17) and (2.18), we conclude

Theorem 2.10. *In* $G\mathcal{B}K - RF_n$ *the curvature vector* H_k *coincides with the tensor* $K_{jk}y^j$ *and the curvature scalar* H *is proportional to the tensor* $K_j y^j$.

3. Conclusions

(3.1) The space whose defined by the condition (2.1) is called generalized $\mathcal{B}$K – recurrent space .

(3.2) In generalized $\mathcal{B}$K – recurrent space , Berwald covariant derivative of first order for the the h(*v*) - torsion tensor H^i_{kh} and the deviation tensor H^i_h and the associate curvature K_{ijkh} given by (2.2),(2.3) and(2.10), respectively.

(3.3) In generalized $\mathcal{B}$K – recurrent space, K-Ricci tensor K_{jk} ,the curvature vector K_j the curvature vector H_k (in the sense of Berwald) and the curvature scalar H are non – vanishing.

(3.4) In generalized $\mathcal{B}$K – recurrent space, the tensor K^s_{skh} , $C_{ijs}H^s_{hk}$ and $H_{.rkh}y^r$ behaves as recurrent.

(3.5) In generalized $\mathcal{B}$K – recurrent space, we have the identities (2.14) ,(2.15), (2.16), (2.17) and (2.18) .

(3.6) In generalized $\mathcal{B}$K – recurrent space, the curvature vector H_k coincides with the tensor $K_{jk}y^j$ and the curvature scalar H is proportional to the tensor $K_j y^j$.

4. Recommendation

Authors recommend the need for the continuing research and development in generalized $\mathcal{B}$K – recurrent Finsler space by obtaining the necessary and sufficient condition the certain tensors to be generalized recurrent in such space.

References

[1] C. K. Misra and G. Lodhi " On C^h -Recurrent and C^v -Recurrent Finsler Space of Second Order" , Int. J. Contemp. Math. Sciences, Vol. 3, No. 17, (2008), 801-810.

[2] F. Y. A. Qasem" On Transformation in Finsler Spaces", D.Phil. Thesis, University of Allahabad, (Allahabad), (India), (2000).

[3] F. Y. A. Qasem and M. A. A. Ali: Certain Types K^h-Birecurrent Finsler Spaces, Journal A bhath, Faculty of Education, (Hodiedah), (Yemen), Volume (2), Issue(3), (January 2015), 21-37.

[4] F. Y. A. Qasem and A. M. A. Hanballa" On Study K^h-Generalized Birecurrent Finsler Spaces", International Journal of Sciences :Basic and Applied Research, Volume 25, No. 3, (2016), 208-216.

[5] H. Rund: The differential geometry of Finsler spaces, Springer-Verlag, Berlin Göttingen-Heidelberg, (1959); 2nd Edit. (in Russian), Nauka, (Moscow), (1981).

[6] M. A. A. Ali" On K^h-Birecurrent Finsler Spaces", M. Sc. Thesis, University of Aden, (Aden), (Yemen), (2014).

[7] M. A. H. Alqufail, F. Y. A. Qasem and M. A. A. Ali" On Study K^h-Birecurrent Finsler Spaces", the Scientific Journal of the Faculty of Education- Thamar, University of Thamar, (Thamar), (2015).

[8] M. Matsumoto" On C-reducible Finsler Space",Tensor N. S., 24, (1972), 29-37.

[9] N. S. H. Hussien" On K^h-Recurrent Finsler Space", M. Sc. Thesis, University of Aden, (Aden), (Yemen), (2014).

[10] S. Dikshit: Certain Types of Recurrences in Finsler Spaces, D.Phil. Thesis, University of Allahabad, (Allahabad) (India), (1992).

[11] P. N. Pandey, S. Saxena and A. Goswani"On Generalized H- Recurrent Finsler Space", Journal of International Academy of Physical Sciences, Vol. 15, (2011), 201-211.

[12] R. Verma: Some Transformation in Finsler Spaces, D.Phil. Thesis, University of Allahabad, (Allahabad) (India), (1991).