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Positive $P_{0,1}$ Matrix Completion Problem for Digraphs of Order 3 with 4 and 5 Arcs

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Abstract

In this paper the positive $P_{0,1}$ matrix completion problem for digraphs of order 3 with 4 and 5 arcs is considered. It is shown that partial positive $P_{0,1}$ matrices specifying digraphs of order 3 with 4 arcs has completion if it is a clique, otherwise there is no completion. It is further shown that an order 3 digraph which omits only one arc lacks positive $P_{0,1}$ completion.

Keywords: Partial matrix; submatrix; principal minor; matrix completion; positive $P_{0,1}$ matrix; symmetric pattern; clique; isomorphism.

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1. Introduction

A partial matrix is a rectangular array in which some entries are specified while others are free to be chosen. A completion of a partial matrix is a specific choice of values for the unspecified entries. A pattern for an $n \times n$ matrix is a list of positions of an $n \times n$ matrix which is a subset of the $\{1,2,\ldots,n\} \times \{1,2,\ldots,n\}$. A partial matrix specifies a pattern if its specified entries lie exactly in those positions listed in the pattern [1].

A **submatrix** of an $n \times n$ matrix A is a matrix obtained from A by deleting some rows and columns of A. For $\propto$ a subset of $\{1,2,\ldots,n\}$, a principal submatrix $A(\propto)$ of A is obtained from A by deleting all rows and columns of A that are not in $\propto$. The determinant of such a principal submatrix $A(\propto)$ is called the principal minor of $A(\propto)$ [1].

The **matrix completion problem** is concerned with determining whether or not a completion of a partial matrix exists within a certain class of matrices. As such, a description of circumstances is sought in which choices for the unspecified entries may be made from a set so that the resulting matrix, called a completion of the matrix is of the desired type [2].

A real $n \times n$ matrix is called a **positive $P_{0,1}$ matrix** if all its principal minors are nonnegative and all its entries are positive. A partial positive $P_{0,1}$ matrix is a partial matrix in which all fully specified principal submatrices are positive $P_{0,1}$ matrices.

A **graph** $G = (V, E)$ is a finite set of positive integers V whose members are called vertices, and a set E of unordered pairs $\{v, u\}$ of vertices, called edges. A subgraph of the graph G is a graph $H = (V', E')$ where V' is a subset of V and E' is a subset of E [3].

A **digraph** $G = (V, A)$ is a finite set V, of positive integers whose members are called vertices and a set A of ordered pairs (v, u) of vertices called arcs (also called directed edges). A digraph $H = (V', A')$ is said to be a subdigraph of G if every vertex of H is also a vertex of G and every arc of H is also an arc of G [3].

Let G be a digraph, a path that begins and ends at the same vertex is called a **cycle**. A digraph that does not contain any cycles is called an acyclic digraph. A subdigraph of a digraph is called a **clique** if it contains at least three vertices and for each pair of vertices v_i and v_j in the subset, both $v_i \rightarrow v_j$ and $v_j \rightarrow v_i$ exist [4].

Graph theory has played an important role in the study of matrix completion problems. A **positionally symmetric** pattern for $n \times n$ matrices that includes all diagonal positions can be represented by means of a graph $G = (V, E)$ on n vertices and E is the edge set. A **non-symmetric** pattern for $n \times n$ matrices that includes all diagonal positions is best described by means of a digraph $G = (V, A)$ on n vertices.

Since the class of matrices under study are non-symmetric, we use digraphs. A partial matrix that specifies a pattern also specifies the digraph determined by the pattern. The digraphs used in this study include all diagonal positions. Some study has been done on positive $P_{0,1}$ matrix completion. In [4], L. Hogben established that all digraphs of order two have positive $P_{0,1}$ completion. In [5], J. Mutembei established that all digraphs of order three with zero, one, two and three arcs have positive $P_{0,1}$ completion. However, much of the study on this class of matrices has not been done to solve the class of matrices.

In this paper, the positive $P_{0,1}$ matrix completion problem is discussed for order three digraphs with four and five arcs. Throughout this paper, the entries of a partial matrix A is denoted as follows: d_i denotes a specified diagonal entry, a_{ij} denotes a specified off-diagonal entry, and x_{ij} an unspecified entry, $1 \leq i, j \leq n$.

In many situations it is convenient to permute entries of a partial matrix. A **permutation matrix** P is obtained by interchanging rows on the identity matrix, therefore all its entries are either 0 or 1 and there is exactly one 1 in each row and each column of the matrix. A permutation similarity of A is a product PAP^T, where P is a permutation matrix. This is represented on the digraph by renumbering the vertices. As a result of the following Lemma, we can permute a partial positive $P_{0,1}$ matrix and hence renumber digraph vertices as convenient.

Lemma 1.1 [1]. *The class of positive $P_{0,1}$ matrices is closed under permutation.*

Two digraphs G_1 and G_2 are said to be **isomorphic** if it is possible to obtain one digraph from the other by renumbering the vertices.

In the next section all order three digraphs with **4** and **5** arcs are considered and 3×3 matrices specifying these digraphs constructed. The construction of the digraphs is guided by the graphs of order three as given in [6]. Throughout the paper, $\boldsymbol{p}$ represents the number of vertices and $\boldsymbol{q}$ represents the number of arcs of the digraph.

Case 1: $p = 3, q = 4$

The only digraphs in this category up to isomorphism are as follows;

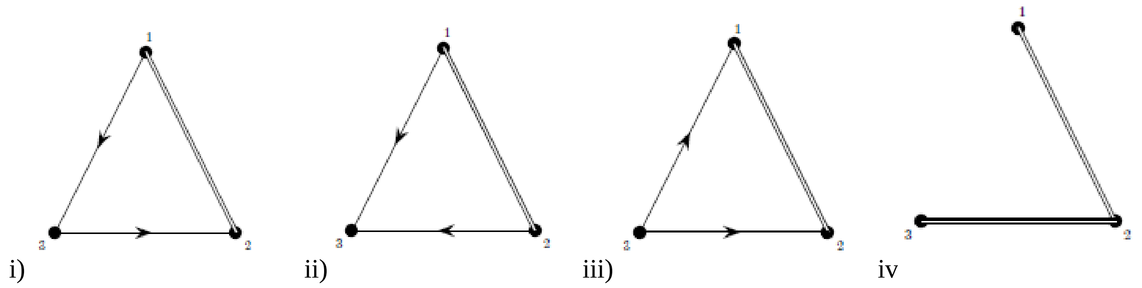

Figure 1

Sub case (i)

Let $A = \begin{pmatrix} d_1 & a_{12} & a_{13} \\ a_{21} & d_2 & x_{23} \\ x_{31} & a_{32} & d_3 \end{pmatrix}$ be the partial positive $P_{0,1}$ matrix specifying the digraph (i) above. By definition of the completion $d_1 > 0, d_2 > 0, d_3 > 0, a_{12} > 0, a_{13} > 0, a_{21} > 0, a_{32} > 0$.

Considering the principal minors of A;

Table 1

Submatrix	Determinant
$A(1,2)$	$d_1d_2 - a_{12}a_{21}$
$A(1,3)$	$d_1d_3 - a_{13}x_{31}$
$A(2,3)$	$d_2d_3 - a_{32}x_{23}$
$A(1,2,3)$	$d_1d_2d_3 + a_{12}x_{23}x_{31} + a_{13}a_{32}a_{21} - a_{13}d_2x_{31} - x_{23}a_{32}d_1 - d_3a_{21}a_{12}$

By definition of the completion $a_{13}x_{31} \leq d_1d_3$, and $a_{32}x_{23} \leq d_2d_3$.

Choosing $a_{13}x_{31} = d_1d_3$ and $a_{32}x_{23} = d_2d_3$ gives;

$$|A| = d_1d_2d_3 + a_{12}x_{23}x_{31} + a_{13}a_{32}a_{21} - d_1d_2d_3 - d_2d_3d_1 - d_3a_{21}a_{12}$$

$$= a_{12}x_{23}x_{31} + a_{13}a_{32}a_{21} - d_1d_2d_3 - d_3a_{21}a_{12}$$

Setting $x_{23} = \frac{d_2d_3}{a_{32}}$ and $x_{31} = \frac{d_1d_3}{a_{13}}$;

$$|A| = \frac{a_{12}d_1d_2d_3^2}{a_{32}a_{13}} + a_{13}a_{32}a_{21} - d_1d_2d_3 - d_3a_{12}a_{21}$$

$$= \frac{(d_1d_2d_3 - a_{21}a_{32}a_{13})(a_{12}d_3 - a_{13}a_{32})}{a_{32}a_{13}}$$

The expression for $|A|$ is inconclusive. Taking a counter example, let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & x \\ y & 3 & 2 \end{pmatrix}$

$$|A| = 4 + xy + 3 - 2y - 3x - 2$$

$$= 5 + xy - 2y - 3x$$

Setting $x = \frac{4}{3}$ and $y = 2$ and simplifying gives; $|A| = -\frac{1}{3} < 0$.

Hence it is impossible to get x_{23} and x_{31} that complete A. Therefore there is no completion for the matrix, hence the digraph has no completion.

By a similar argument, it can be shown that the partial matrices specifying the digraphs (ii) and (iii) have no completion.

Sub case (ii)

Let $A = \begin{pmatrix} d_1 & a_{12} & x_{13} \\ a_{21} & d_2 & a_{23} \\ x_{31} & a_{32} & d_3 \end{pmatrix}$ be the partial positive $P_{0,1}$ matrix representing the digraph (iv) above. By definition of the completion $d_1 > 0, d_2 > 0, d_3 > 0, a_{12} > 0, a_{21} > 0, a_{23} > 0, a_{32} > 0$. Considering the principal minors of A;

Table 2

Submatrix	Determinant
$A(1,2)$	$d_1 d_2 - a_{12} a_{21}$
$A(1,3)$	$d_1 d_3 - x_{13} x_{31}$
$A(2,3)$	$d_2 d_3 - a_{23} a_{32}$
$A(1,2,3)$	$d_1 d_2 d_3 + a_{12} a_{23} x_{31} + x_{13} a_{32} a_{21} - x_{13} d_2 x_{31} - a_{23} a_{32} d_1 - d_3 a_{21} a_{12}$

By definition of the completion $x_{13} x_{31} \leq d_1 d_3$ .Choosing $x_{13} x_{31} = d_1 d_3$ gives;

$$|A| = d_1 d_2 d_3 + a_{12} a_{23} x_{31} + x_{13} a_{32} a_{21} - d_1 d_2 d_3 - a_{23} a_{32} d_1 - d_3 a_{21} a_{12}$$

$$= a_{12} a_{23} x_{31} + x_{13} a_{32} a_{21} - a_{23} a_{32} d_1 - d_3 a_{21} a_{12}$$

Setting $x_{13} = \frac{d_1 d_3}{x_{31}}$ gives;

$$|A| = a_{12} a_{23} x_{31} + a_{21} a_{32} \frac{d_1 d_3}{x_{31}} - a_{23} a_{32} d_1 - d_3 a_{21} a_{12}$$

$$= \frac{a_{12} a_{23} x_{31}^2 + a_{21} a_{32} d_1 d_3 - a_{23} a_{32} d_1 x_{31} - d_3 a_{21} a_{12} x_{31}}{x_{31}}$$

$$= \frac{(d_3 a_{21} - a_{23} x_{31})(d_1 a_{32} - a_{12} x_{31})}{x_{31}}$$

By choosing the unspecified entry x_{31} to be either very large or very small, $|A| > 0$, hence there is completion for the partial matrix A. Therefore the partial matrix specifying the digraph (iv) above has completion.

Case 2: $\boldsymbol{p = 3, q = 5}$

The only digraph in this case up to isomorphism is;

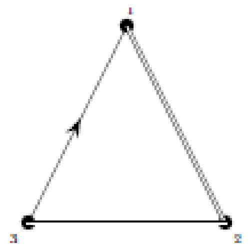

Figure 2

Let $A = \begin{pmatrix} d_1 & a_{12} & x_{13} \\ a_{21} & d_2 & a_{23} \\ a_{31} & a_{32} & d_3 \end{pmatrix}$ be the partial positive $P_{0,1}$ matrix specifying the digraph above. By definition of the completion $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, $a_{12} > 0$, $a_{21} > 0$, $a_{23} > 0$, $a_{32} > 0$. Considering the principal minors of A;

Table 3

Submatrix	Determinant
$A(1,2)$	$d_1d_2 - a_{12}a_{21}$
$A(1,3)$	$d_1d_3 - x_{13}a_{31}$
$A(2,3)$	$d_2d_3 - a_{23}a_{32}$
$A(1,2,3)$	$d_1d_2d_3 + a_{12}a_{23}a_{31} + x_{13}a_{32}a_{21} - x_{13}d_2a_{31} - a_{23}a_{32}d_1 - d_3a_{21}a_{12}$

By definition of the completion $x_{13}a_{31} \leq d_1d_3$. Choosing $x_{13}a_{31} = d_1d_3$ gives;

$$|A| = d_1d_2d_3 + a_{12}a_{23}a_{31} + x_{13}a_{32}a_{21} - d_1d_2d_3 - a_{23}a_{32}d_1 - d_3a_{21}a_{12}$$

$$= a_{12}a_{23}a_{31} + x_{13}a_{32}a_{21} - a_{23}a_{32}d_1 - d_3a_{21}a_{12}$$

Setting $x_{13} = \frac{d_1d_3}{a_{31}}$,

$$|A| = a_{12}a_{23}a_{31} + a_{32}a_{21}\frac{d_1d_3}{a_{31}} - a_{23}a_{32}d_1 - d_3a_{21}a_{12}$$

$$= \frac{a_{12}a_{23}a_{31}^2 + a_{32}a_{21}d_1d_3 - a_{23}a_{32}d_1a_{31} - d_3a_{21}a_{12}a_{31}}{a_{31}}$$

$$= \frac{(a_{23}a_{31} - a_{21}d_3)(a_{31}a_{12} - d_1a_{32})}{a_{31}}$$

This expression is inconclusive. Taking a counter example, let $A = \begin{pmatrix} 5 & 2 & x \\ 4 & 2 & 2 \\ 5 & 1 & 3 \end{pmatrix}$.

$$|A| = 30 + 20 + 4x - 10x - 10 - 24$$

$$= 16 - 6x$$

Setting $x = 3$ gives $|A| = -2 < 0$.

Hence it is impossible to find x_{13} which completes A .Therefore there is no completion for the partial matrix A which specifies the digraph above.

1. Conclusions

The results in this study show that an order three digraph with 4 arcs, which is not a clique does not have positive $P_{0,1}$ completion. It has also been shown that a digraph of order 3 which omits only one arc does not have positive $P_{0,1}$ completion. This study may be extended further to higher order matrices in order to solve this class of matrices.

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