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On the Expansion of a Spacings Based Statistics

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Abstract

The statistics based on gaps between points usually termed as spacings has been extensively studied in many contexts particularly for testing the hypothesis. When the exact distribution is either unavailable or does not exist in tractable form then it is useful to approximate the distribution of such statistics. One of the most famous among others is the Edgeworth Expansion providing such services. There is a huge literature devoted to studying the distribution of random variable based on uniform Spacings. One of them is the Log- spacings Statistics. We aim to approximate the distribution of Log- spacing statistics by Edgeworth type expansion.

Keywords: Spacings; i.i.d Random Variable; Uniform spacings; Edgeworth expansions.

1. Introduction

Let $X_1, X_2, ..., X_{n-1}$ be an ordered (ascending form) sample from a population having continuous cumulative distribution function (cdf) F and probability density function (pdf) $f(x)$. The goodness-of-fit problem is to test if this distribution is equal to a specified one. A common approach to these problems is to transform the data via the probability integral transformation $U = F(X)$ so that the support of F is reduced to [0,1] and the specified cdf reduces to that of a uniform random variable on [0,1].

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With notations $X_0 = 0$ and X_{n+1}, sample spacings are defined as $G_j = X_j - X_{(j-1)}$, $j = 1, 2, ..., n.$ Let $f : [0, \infty) \to R$ be a fixed non linear measurable function and define the statistics based on spacings as

$$R_n = \sum_{j=1}^{n} f\left(nG_j\right), \qquad n = 1, 2, ... \qquad (1.1)$$

When $f(x,u) = f(u)$ then the random variable R_n is symmetric. The statistics R_n is extensively studied by the researchers for different $f(x)$ as kernel function. For example, the statistics is called linear when $f(x, y) = l(y)x$ where $l(y)$ is real function defined on [0,1], see for example, [11,12,22,25,28]. When $f(x) = x^2$, it is the classical case of Greenwood statistics (cf.[10]).The development of goodness-of-fit tests based on uniform spacings received its principal impetus after the introduction of Greenwood statistics. The statistics is called log-spacings statistic if $f(x) = \log x$, see ,for example, [3,5,25]. The random variable R_n is entropy-type spacings statistics if $f(x) = x \log x$, see for instance, [1, 13, 15]. The random variable R_n is generalized Rao's statistic when $f(x) = |x - k|^r$, $r > 0$,the case $r = 2$ coincide with Greenwood statistic [10], and $r = 1$ is Rao's spacings statistic see [31] , see also[6,22]. The random variable is used for circular data when $f(x) = \sin x$, see for example, [26,27,29]. The random variable R_n is called Extreme-spacings statistic when $f(x) = \{1(x \le a_n) + 1(x \ge b_n)\}$;where $1(A)$ is indicators function, a_n and b_n are constants see, for example, [14,23]. A number of statistical problems related to the distribution of $f(x) = \{1(x \le a_n) + 1(x \ge b_n)\}$ have been discussed by the author in [3]. The research article in [3, 4, 13] provides a unified treatment to the distributions of random variable R_n. For the random variable R_n the authors of [21] obtained an estimation of the remainder term in CLT, also it is worth noticing that the probability of large deviations of random variable R_n, a problem less investigated earlier, is proved by [20]. Sometimes the exact distribution of the random variable in tractable form is not available. Even if it exists often its rate of convergence to the normal form is very slow see, for example, [8]. That is the reason the researchers has shown considerable interest into the asymptotic distribution theory for the statistics of type (1.1). As compared to normal approximation in which only the mean and variance play a role, the Edgeworth approximation is worth noticing as involves the first four moments of the statistics. For this reason, sometimes better approximations for the distribution function of (1.1) may be obtained easily by using Edgeworth expansions. The advantage of the Edgeworth series is that the error is controlled, so it is a true asymptotic expansion. Some authors calculated Edgeworth expansion of spacing statistics for small to moderate sample sizes; see for example, [9]. The Edgeworth series approximation for large sample sizes is available in many articles see for example, [17,27]. The validity of formal Edgeworth expansions has been proved under suitable assumptions in [2]. The famous research article given in [7] has established Edgeworth expansions for statistics (1.1) under a natural moment assumption and an appropriate version of Cramer- type condition. They have shown that a Cramer-type condition holds under an easily verifiable and mild assumption on the function f : if (c,d), contained in $(0,\infty)$, is an open interval on which " f "is almost everywhere differentiable and the derivative of " f "is not essentially constant on the prescribed interval then the Cramer-type condition is satisfied. By using the characterization of [7], we aim to find the Edgeworth type expansion for random variable R_n when $f(x)=\log x$. The paper is organized as, in section (Edgeworth type Expansion) we formulate our theorem, in section (Asymptotic Normality) we discuss the limit theorem for our statistics, in section (Preliminary Lemmas) we recall some preliminary results and state two lemmas (without proof) necessary for the proof of our result, in section (Proof of Theorem-1) we will prove our result.

2. Asymptotic Normality

The asymptotic normality and Cramers type large deviation theorem for statistics (1.1) have been obtained in [19] and [24] for the sum of functions of uniform spacings (i.e. under null hypothesis). Note that under alternatives converging to uniform null hypothesis the spacings D_j can be reduced to uniform spacings for details see [22]. Since $Y_1,Y_2,...,Y_n$ are exponential random variables with expectation 1 and " f "is a fixed real- valued measurable function defined on R^+. We suppose that the moments used below exist.

Let $\rho = corr(R_n(Y),S_n)$, $g_j(u)=f_j(u)-Ef_j(u)-(u-1)\rho\sqrt{\frac{\operatorname{var}R_n(Y)}{n}}$, $H_n(D)=\sum_{j=1}^{n} g_j(nD_j)$

$H_n(Y)=\sum_{j=1}^{n} g_j(Y_j)$, $A_n=\sum_{j=1}^{n} Ef_j(Y_j)$. Note that $\sum_{j=1}^{n} Eg_j(Y_j)=0$ and $\beta_{mn}=\sum_{j=1}^{n} E\left|g(Y_j)/\sigma_n\right|^m$

$\sigma_N^2 = VarH_n(Y)=\left(1-\rho^2\right)Var\left(R_n\left(Y\right)\right)$, $EH_n(Y)=0$ also $Cov\left(H_n\left(Y\right),S_n\right)=0$. It is obvious that $H_n\left(D\right)=R_n\left(D\right)-E\left(R_n\left(Y\right)\right)$. Therefore, without loss of generality, we may consider the statistics $H_n\left(D\right)$ instead of $R_n\left(D\right)$. From definition of σ_n^2 follows that $\sigma_n^2=0$ if and only if $f_j(u)=Cy+\lambda_j$, j=1,2,…,n ,where constants λ_j are arbitrary and C does not depend on m for all j=1,…,n. We suppose that $\sigma_n^2>0$ for all n=1,2,… .We note that $f_j\left(Y\right)$ are random functions. In such a case $f_j\left(Y\right), j=1,2,...,n$ is a sequence of independent random variables not depending on D or Y.

Lemma 2-1. If $\beta_{3,n}\to 0$ as $n\to\infty$ then the random variable R_n has asymptotical normal distribution with expectation A_n and variance σ_n^2.

The Lemma presented above is the corollary 2 of [19].

3. Edgeworth Expansion.

Consider a sample $U_1, U_2, ..., U_{n-1}$ from the uniform [0, 1] distribution with $0 = U_0 \le U_1 \le ... \le U_{n-1} \le U_n = 1$ its order statistics and $D_j = U_j - U_{(j-1)}$ is their uniform spacings.

Then the random variable

$$L_n = \sum_{j=1}^{n} Log\left(nD_j\right) \ , \qquad (3.1).$$

is a special case of (1.1) with kernel function $f(x) = Log(x)$. We will derive Edgeworth type Expansion for (3.1). Clearly D_j , $j = 1, 2, ...$ are simple uniform spacings. Let Y_j, $j = 1, 2, ..., n$ be exponential random variables distributed identically and independently (i.i.d) with mean 1 and $\Im\left(X\right)$ denote distribution of a random vector X. Then it is well known the $\Im\left(D_1, D_2, ..., D_n\right) = \Im\left(Y_1, Y_2, ..., Y_n \Big/ \sum_{j=1}^{n} Y_j = n\right)$

(see, for example, [18]). Let $\Phi\left(x\right)$ be the standard normal distribution and $\phi\left(x\right) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ be the normal density.

$$\Psi_n(x) = \Phi(x) - \phi\left(x\right)\left[n^{\frac{-1}{2}}\left\{-\frac{1}{6}\left(\frac{2711}{1000}\right)\left(x^2 - 1\right) + \frac{3113}{5000}\right\} + n^{-1}\left\{\frac{1}{24}\left(\frac{1029}{50}\right)\left(x^3 - 3x\right) + \left(\frac{1}{72}\right)\left(-\frac{2711}{1000}\right)^2 \left(x^5 - 10x^3 + 15x\right) + \frac{1}{8}\left(-4\left(\frac{3113}{5000}\right)\left(-\frac{2711}{1000}\right) + \left(\frac{49}{20}\right)\right) x + \left(\frac{1}{6}\left(\frac{3113}{5000}\right)\left(-\frac{2711}{1000}\right)\right)x^3\right\}\right]$$

or we can write

$$\Psi_n\left(x\right) = \Phi(x) - \phi\left(x\right)\left[n^{-\frac{1}{2}}\left\{\frac{2711}{6000}\left(1 - x^2\right) + \frac{3113}{5000}\right\} + n^{-1}\left\{\left(\frac{343}{400}\right)\left(x^3 - 3x\right) + \frac{51}{500}\left(x^5 - 10x^3 + 15x\right) + \left(\frac{23}{20}\right)x - \frac{281}{1000}x^3\right\}\right] \qquad (3.2)$$

Theorem 3-1.Let $\nabla_n = n - L_n$ with $\Psi_n\left(x\right)$ as in (3.2) while $\mu = E\left(f\left(x\right)\right) = \gamma \approx 0.5772...$ and $\sigma_n^2 = Var\left(f\left(x\right)\right) = \zeta\left(2\right) \approx 1.6449...$.Then $P\left\{\left(\nabla_n - \mu\right)\sigma_n^{-1} \le x\right\} = \Psi_n\left(x\right) + o\left(1/n\right)$, x ε R as $n \to \infty$.

The statistic L_n is a special case of (1.1) with $f(u) = \log u$. Therefore, by the Lemma 2-1 we get

Theorem 3-2. The Statistics L_n has asymptotically normal distribution with expectation nA and variance $n\sigma^2$ as $n \to \infty$. Here $A = \gamma$ and $\sigma^2 = \zeta(2)$.

Proof. It is obvious that $E\left(f^2(u)\right) < \infty$, so by the assertion stated above statistic L_n is asymptotically normal with parameters $nE\left(f(u)\right)$ and $n\left(1-\rho^2\right)Var\left(f(u)\right)$ where ρ is the correlation between *f*(*u*) and *u*. By direct calculation it is easy to find $A = E\left(f(u)\right), \left(1-\rho^2\right)$ and $\sigma^2 = Var\left(f(u)\right)$.

4. Preliminary lemmas

The aim of this section is to settle some technical points. Only statements of the required lemmas are presented here. For proof one should see the paper [7].

Lemma 4-1: Let $g:[0,\infty) \to R$ be a non linear measurable function whose derivative exists and is not necessarily constant on $(c,d) < (0,\infty)$ such that E g^4 (y) <∞. Define the statistics

$T_n = \sum_{j=1}^{n} g\left((n+1)D_{j,n}\right), n = 1,2,...$ If F_n is the distribution of $\left(T_n - ET_n\right)/\sqrt{\operatorname{var} T_n}$ and Ψ_n is as in (3.2) then $\lim_{n\to\infty} n \sup_{x\in R} \left|F_n(x) - \Psi_n(x)\right| \le o(1)$.

Lemma 4-2: Let x be a random variable taking values in R^m , the distribution of which is absolutely continuous on some Boral set B with $P(X \in B) > 0.$ Let $f: R^m \to R^k$ be a measurable function which is Lebesgue almost everywhere differentiable on B with the k x m matrix f as differential. If all $\chi \in R^k \setminus \{0\}$ satisfy $P\left\{\left(h(X)\right)^T \chi = 0 / X \in B\right\} < 1$. Then

$$\underset{|\chi|\to\infty}{Lim} \sup \left|Ee^{i\chi^T h(X)}\right| < 1 \text{ Holds.}$$

5. Proof of Theorem 3 -1

By basic definition the gamma function is given by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ .Since the integrand is of exponential family so it is continuous and by the well known fact the derivatives of all order for this function exists. Thus for $n \geq 0$ one can write $\frac{d^n}{dx^n}\Gamma(x) = \int_0^\infty t^{x-1}(\log t)^n e^{-t} dt.$ Also, for even n zeta functions are defined as

$\zeta(n) = 2^{n-1}|B_n|\pi^n(n!)^{-1}$ see, for example, [5]. Particularly, $\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90},\ldots\ B_n$ are the well known Bernoulli numbers with $B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30}$,.... although, no analytic form for $\zeta(n)$ is known for odd n but the one we need here is $\zeta(3) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k}{k^2} \approx 1.2020569032....$

Where $H_n = \gamma + \psi(n+1)$ are harmonic numbers with $\psi(n)$ is a digamma function a

$\gamma = 0.577215664...$ is Euler constant.

It is to be noted that all the conditions settled in the two lemmas given in section-4 are satisfied by the statistics given in (3.1).

Therefore, if $\tilde{\Psi}_n$ is the distribution of $\tilde{L}_n = (L_n - EL_n)\big/\sqrt{Var(L_n)}$ and $\Psi_n(x)$ is as in (3.2), then

$$\lim_{n\to\infty} n \sup_{x\in R}\left|\tilde{\Psi}_n(x) - \Psi_n(x)\right| = O(1).$$

We replace $f(Y)$ by $\tilde{f}(Y) = \left(f(Y) - \mu - \tau(Y-1)\right)\left(\sigma^2 - \tau^2\right)^{-1/2}$ which is merely a sort of centralization and does not affect the distribution of $\tilde{\Psi}_n$. We get different parameters as

$$\mu = \gamma,\quad \sigma^2 = \zeta(2) = \frac{\pi^2}{6},\quad \tau^2 = 1,$$

$$\kappa_3 = E\left(\tilde{f}(Y)\right)^3 = \{1-\zeta(2)\}^{-3/2}\{1-2\zeta(3)\} \approx -2.710998846...$$

$$\alpha = -\frac{1}{2}E\left(\tilde{f}(Y)(Y-1)^2\right) = -\frac{1}{2}\{\zeta(2)-1\}^{-1/2} \approx 0.622604629...$$

$$\begin{aligned}\kappa_4 &= E\left(\tilde{f}(Y)\right)^4 - 3 - \left(E\ \tilde{f}^2(Y)(Y-1)\right)^2 \\ &= \{1-\zeta(2)\}^{-2}\left\{\frac{27}{2}\zeta(4) + 8\zeta(3) - 8\gamma\zeta(3) - 6\zeta(2) + 1\right\} - 3 \approx 20.57899468...\end{aligned}$$

$$\beta = 3\left(E\ \tilde{f}(Y)(Y-1)^2\right)^2 - 2E\ \tilde{f}^2(Y)(Y-1)^2 + 4E\ \tilde{f}(Y)(Y-1)^3 + 6$$
$$= 3\{\zeta(2)-1\}^{-1} - 2\{\zeta(2)+1\} + 6 \approx 2.449453903...$$

So that the Edgeworth type expansion $\Psi_n(x)$ of function $\tilde{f}$ is as given in (3.2). Note that $E\left(f^4(Y)\right) = 553/20 < \infty$ so that the first condition of Lemma 4-1 is satisfied.

By taking m = 1, k = 1, $h(x) = \left(x, \dfrac{f(x)-x+1.5772}{\sqrt{\zeta(2)-1}}\right)$ and B = (0, ∞) in Lemma 4-2 and let $\chi = (c,d)$ then $(h(x))^T \chi = \left[x \quad \dfrac{f(x)-x+1.5772}{\sqrt{\zeta(2)-1}}\right]\left[c \quad d\right]^T = cx + \dfrac{f(x)-x+1.5772}{\sqrt{\zeta(2)-1}}d$.

For $(h(x))^T.\chi = 0$, three cases arises

(i) c = 0, d≠0 , (ii) c≠0, d=0 , (iii) c ≠ 0, d≠0. For all the three possible cases

$$P\left\{(h(x))^T.\chi = 0 / x \in B\right\}.$$

So if Q (s, t) is the characteristic function of $(Y, f(Y))$ then by lemma 4-2

$\lim\limits_{(s,t)\to\infty}$ Sup | Q (s, t) | < 1 also $\dfrac{d}{dy}(Log\ y) = \dfrac{1}{y}$ is not constant on $(0,\infty)$.Hence by lemma 4-1 $\lim\limits_{n\to\infty}$ n sup $\left|F_n(x) - \Psi_n(x)\right| = o(1)$, x ε R Where F_n is the distribution of $(\nabla_n - \mu)\sigma_n^{-1}$ that is $\lim\limits_{n\to\infty} p\left\{(\nabla_n - \mu)\sigma_n^{-1} \le x\right\} = \Psi_n(x) + o\left(\dfrac{1}{n}\right)$. This completes the proof.

The Edgeworth expansion is calculated using Mathematica for n=10, 20, 30, 50, 70, 100, 250, 300,500 and 11000 in the region $|x| \le 3$. In the following table the Edgeworth expansion i.e. Ψ_n is tabulated for the above mentioned sample sizes and different values of the arguments.

Table 1

x	**-3**	**-2.5**	**-2**	**-1.5**	**-1**	**-.5**	**0**	**.5**	**1**	**1.5**	**2**	**2.5**	**3**
Ψ_{10}	**.007**	**.008**	**-.003**	**-.036**	**-.067**	**-.049**	**.367**	**.333**	**.626**	**.845**	**.951**	**.985**	**.995**
Ψ_{20}	**.004**	**.007**	**.004**	**-.004**	**.000**	**.056**	**.433**	**.375**	**.688**	**.869**	**.959**	**.989**	**.997**
Ψ_{30}	**.003**	**.006**	**.008**	**.010**	**.030**	**.102**	**.456**	**.485**	**.716**	**.880**	**.962**	**.990**	**.998**
Ψ_{50}	**.002**	**.006**	**.011**	**.023**	**.059**	**.149**	**.473**	**.532**	**.743**	**.892**	**.965**	**.992**	**.998**
Ψ_{70}	**.002**	**.006**	**.013**	**.030**	**.075**	**.173**	**.481**	**.556**	**.758**	**.898**	**.967**	**992**	**.999**
Ψ_{100}	**.002**	**.006**	**.015**	**.036**	**.089**	**.196**	**.487**	**.579**	**.772**	**.904**	**.969**	**.992**	**.999**
Ψ_{150}	**.002**	**.006**	**.016**	**.042**	**.102**	**.216**	**.491**	**.599**	**.784**	**.909**	**.970**	**.993**	**.999**
Ψ_{250}	**.001**	**.006**	**.018**	**.048**	**.115**	**.237**	**.394**	**.620**	**.797**	**.914**	**.972**	**.993**	**.999**
Ψ_{300}	**.001**	**.006**	**.018**	**.049**	**.119**	**.244**	**.496**	**.627**	**.801**	**.916**	**.972**	**.993**	**.999**
Ψ_{500}	**.001**	**.006**	**.019**	**.053**	**.128**	**.258**	**.497**	**.641**	**.810**	**.920**	**.973**	**.993**	**.999**
Ψ_{11000}	**.001**	**.006**	**.022**	**.064**	**.152**	**.298**	**.500**	**.681**	**.834**	**.930**	**.976**	**.994**	**.999**
Φ	**.001**	**.006**	**.023**	**.067**	**.159**	**.309**	**.500**	**.692**	**.841**	**.933**	**.977**	**.994**	**.999**

6. Conclusion

From the table we observe that although the conditions used by [7] are not strong enough even then Edgeworth type Expansion obtained by his method perform very well. His method is not hard and can be easily applied. The convergence to normal form is considerably fast.

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