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On Certain Types of Affine Motion

Fahmi Y. A. Qasem[a*], Abdalstar A. M. Saleem[b]

[a]*Department of Mathematics , Faculty of Education-Aden, University of Aden, Khormaksar , Aden, Yemen*

[b]*Department of Mathematics , Faculty of Education-Yafea, University of Aden*

[a]*Email: Fahmiyassen@gmail.com*

[b]*Email: Abdulsstar@yahoo.com*

Abstract

In the present paper, the affine motion and the projective motion generated by recurrent in a general Finsler space is studied, the necessary and sufficient conditions for this projective motion to be affine motion are obtained. projective motion is studied in recurrent Finsler space.

Keywords: Finsler space; affine motion; projective motion; hv-curvature tensor U^i_{jkh}; U- recurrent space; U-birecurrent space; projective recurrent space.

1. Introduction

K. Takano and T. Imai [15] studied certain types of affine motion generated by contra, concurrent, special concircular, recurrent, concircular, torse forming and birecurrent vector fields in a non-Riemannian space of recurrent curvature and ended with some remarks on the affine motion in a space with recurrent curvature. K. Takano and T. Imai [15], P. N. Pandey and V. J. Dwivedi [8] further wrote a series of three papers on the existence affine motion in a non-Riemannian space of recurrent curvature and obtained various interesting results. K. Takan and T. Imai [15] and S. P. Singh [14] discussed the affine motion in a birecurrent non-Riemannian space.

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* Corresponding author.

Several results obtained by these authors were extended to Finsler spaces of recurrent curvature by R. B. Misra [6], F. M. Meher [5], A. Kumar ([1], [2], [3]), A. Kumar, H. S. Shukla and R. P. Tripathi [4], P. N. Pandey, F. Y. A. Qasem and Suinta Pal [9], S. P. Singh [13] and others. K. Yano [16] defined the normal projective connection coefficients Π^i_{jk} by

(1.1) $$\Pi^i_{jk} = G^i_{jk} - y^i G^r_{jkr}.$$

The connection coefficients Π^i_{jk} is positively homogeneous of degree zero in y^i 's and symmetric in their lower indices and the normal projective tensor N^i_{jkh} is defined as follows [16]:

(1.2) $$N^i_{jkh} = \Pi^i_{jkh} + \Pi^i_{rjh}\Pi^r_{ks}y^s + \Pi^i_{rh}\Pi^r_{kj} - k|h\ ,$$

where

(1.3) $$\Pi^i_{jkh} = \dot{\partial}_j \Pi^i_{kh}\ .$$

Π^i_{jkh} constitutes the components of a tensor.

Remark 1.1. K. Yano [16] denoted the tensor Π^i_{jkh} by the curvature tensor U^i_{jkh}.

The curvature tensor U^i_{jkh} is defined by

(1.4) $$U^i_{jkh} = G^i_{jkh} - \frac{1}{n+1}\left(\delta^i_j G^r_{jkr} + y^i G^r_{jkhr}\right).$$

is called *hv-curvature tensor*, where G^i_{jkh} is connection of hv-curvature tensor. Also this tensor satisfy the following:

(1.5) $$U^i_{jkh} y^j = 0.$$

We also have the following commutation formulae [12]

(1.6) $$\left(\dot{\partial}_j \mathcal{B}_k - \mathcal{B}_k \dot{\partial}_j\right) X^i = U^i_{jkh} X^h - \left(\dot{\partial}_r X^i\right) U^r_{jkh} y^h$$

and

(1.7) $$\mathcal{B}_k \mathcal{B}_h T^i_j - \mathcal{B}_h \mathcal{B}_k T^i_j = T^r_j \mathrm{N}^i_{rkh} - T^i_r \mathrm{N}^r_{jkh} - (\dot{\partial}_r T^i_j)\mathrm{N}^r_{skh} y^s.$$

A Finsler space is called recurrent Finsler space and birecurrent Finsler space, respectively, denoted them by *UR*-F_n and *UBR*-F_n, respectively, if it's hv- curvature tensor U^i_{jkh} satisfies ([10], [11])

(1.8) $$\mathcal{B}_m U^i_{jkh} = \lambda_m U^i_{jkh}, \qquad U^i_{jkh} \neq 0$$

and

(1.9) $\mathcal{B}_l\mathcal{B}_m U^i_{jkh} = a_{ml} U^i_{jkh}, \qquad U^i_{jkh} \neq 0,$

where λ_m and a_{lm} are non-zero covariant vector and tensor fields.

Let us consider a transformation

(1.10) $\bar{x}^i = x^i + \varepsilon\, v^i(x^j),$

where ε is an infinitesimal constant and $v^i(x^j)$ is called *contravariant vector filed* independent of y^i. The transformation represented by (1.10) is called an *infinitesimal transformation*. Also this transformation gives rise to a process of differentiation called

Lie- differentiation.

Let X^i be an arbitrary contravariant vector filed. Its Lie-derivative with respect to the above infinitesimal transformation is given by ([12], [16])

(1.12) $L_v X^i = v^r \mathcal{B}_r X^i - X^r \mathcal{B}_r v^i + (\dot{\partial}_r X^i)\mathcal{B}_s v^r y^s,$

where the symbol L_v stands for the Lie- differentiation. In view of (1.12), Lie-derivatives of y^i and v^i with respect to above infinitesimal transformation vanish, i.e.

(1.13) $a)\ L_v y^i = 0$

and

$b)\ L_v v^i = 0.$

Lie-derivative an of arbitrary tensor T^i_j with respect to the above infinitesimal transformation is given by

(1.14) $L_v\, T^i_j = v^r \mathcal{B}_r\, T^i_j - T^r_j \mathcal{B}_r v^i + T^i_r \mathcal{B}_j v^r + (\dot{\partial}_r\, T^i_j)\mathcal{B}_s v^r y^s.$

Lie-derivative of the normal projective connection parameters Π^i_{jk} is given by [16]

(1.15) $L_v\, \Pi^i_{jk} = \mathcal{B}_j \mathcal{B}_k v^i - U^i_{rjk} y^s \mathcal{B}_s v^r + N^i_{rjk} v^r.$

The commutation formulae for the operators $\mathcal{B}_k$, $\dot{\partial}_j$ and L_v are given by

(1.16) $(L_v \mathcal{B}_k - \mathcal{B}_k L_v) X^i = X^h L_v \Pi^i_{kh} - (\dot{\partial}_r X^r) L_v \Pi^i_{kh} y^h$

and

(1.17) $\left(\dot{\partial}_j L_v - L_v \dot{\partial}_j\right) X^i = 0.$

where X^i is a contravariant vector filed.

The necessary and sufficient condition for the transformation (1.10) to be a motion, affine motion and projective motion are respectively given by

(1.18) $L_v g_{ij} = 0,$

(1.19) $L_v \Pi^i_{kh} = 0$

and

(1.20) $L_v \Pi^i_{jk} = \delta^i_j P_k + \delta^i_k P_j,$

where P_j is defined as

(1.21) $P_j = \dot{\partial}_j P,$

P being a scalar, positively homogeneous of degree one iny^i.

It is well know that every motion is affine motion and every affine motion is a projective motion. A projective motion need not be affine motion.

2. Affine motion

Let an infinitesimal transformation (1.10) be generated by a vector filed $v^i(x^j)$.The

infinitesimal transformation is an affine motion if and if Lie – derivative of the normal

projective connection parameters Π^i_{jk} with respect to infinitesimal transformation (1.10) vanishes identically, i.e. $L_v \Pi^i_{jk} = 0$.

The vector filed $v^i(x^j)$ is called *contra*, *concurrent*, *special concircular*, *recurrent* and *torse forming* according as it satisfies

(2.1) $a)\ \mathcal{B}_k v^i = 0,$

$b)\ \mathcal{B}_k v^i = c\delta^i_k,$ c being a constant,

$c)\ \mathcal{B}_k v^i = \rho\delta^i_k,$ ρ is not a constant,

$d)\mathcal{B}_k v^i = \mu_k v^i$

and

$e)\ \mathcal{B}_k v^i = \mu_k v^i + \rho\delta^i_k,$

respectively. The affine motion generated by above vectors is called *contra affine motion, concurrent affine motion, special concircular affine motion, recurrent affine motion* and *torse forming affine motion, respectively.*

3. Contra Affine Motion

Let us consider an infinitesimal transformation generated by contra vector $v^i(x^j)$ characterized by (2.1a).

Differentiating (2.1a) covariantly with respect to x^j in the sense of Berwald, we get

(3.1) $\mathcal{B}_j\mathcal{B}_k v^i = 0.$

Taking skew-symmetric part of (3.1), using the commutation formula exhibited by (1.7) for v^i, we get

(3.2) $N^i_{hjk} v^h = 0.$

Using (3.1), (2.1a) and (3.2) in (1.15), we get

(3.3) $L_v \Pi^i_{jk} = 0.$

Hence the infinitesimal transformation considered is an affine motion.

Thus, we conclude

Theorem 3.1. *Every contra vector generates an affine motion in a Finsler space.*

Thus, we conclude

Corollary 3.1. *Every contra vector generates a projective motion.*

Theorem 3.2. *In an $UR - F_n$, if any contra vector $v^i(x^j)$ generates an infinitesimal transformation, it must be orthogonal to the recurrence vector.*

proof

Let us consider an $UR - F_n$ and a contra vector $v^i(x^j)$ characterized by (1.8) and (2.1a), respectively. Then, the hv-curvature tensor U^i_{jkh} satisfies $L_v U^i_{hjk} = 0$.

In view of (1.14) and (2.1a), we get

(3.4) $L_v U^i_{jkh} = v^r \mathcal{B}_r U^i_{jkh}$.

Differentiating (3.3) partially with respect to y^h, we get

(3.5) $\dot{\partial}_h L_v \Pi^i_{jk} = 0$.

Taking skew-symmetric part of (3.5), using the commutation formula exhibited by (1.17) for Π^i_{jk}, (3.5) in view of remark 1.1., we get

(3.6) $L_v U^i_{hjk} = 0$.

In view of (3.4) and (3.6), we get

(3.7) $v^r \mathcal{B}_r U^i_{jkh} = 0$.

Transvecting (1.8) by v^m and using (3.7), we get

(3.8) $v^m \lambda_m = 0$,

where $U^i_{jkh} \neq 0$.

Thus, we see that the contra vector $v^i(x^j)$ is orthogonal to the recurrence vector λ_m.

Theorem 3.3. *In an $UBR - F_n$, if any contra vector $v^i(x^j)$ generates an infinitesimal transformation, then the recurrence tensor a_{lm} satisfies*

(3.9) $a)\ a_{ml} v^m = 0$ and $b)\ v^m a_{lm} = 0$.

proof

Let us consider an $UBR - F_n$ and a contra vector $v^i(x^j)$ characterized by (1.9) and (2.1a), respectively. Then, the hv-curvature tensor U^i_{jkh} satisfies $L_v U^i_{hjk} = 0$.

Differentiating (3.7) covariantly with respect to x^l in the sense of Berwald, using (2.1a) and (1.9), we get

(3.10) $a_{ml} v^m = 0$,

where $U^i_{jkh} \neq 0$, it's the equ. (3.9a).

Taking skew-symmetric part of (3.10), we get

(3.11) $(a_{ml} - a_{lm})v^m = 0$,

where $U^i_{jkh} \neq 0$.

Using (3.9a) in (3.11), we get

(3.12) $a_{lm}v^m = 0$

which its equ. (3.9b).

4. Concurrent Affine Motion

Let us consider an infinitesimal transformation generated by concurrent vector $v^i(x^j)$ characterized by (2.1b).

Differentiating (2.1b) covariantly with respect to x^j in the sense of Berwald, we get (3.1). Taking skew-symmetric part of (3.1), using the commutation formula exhibited by (1.7) for v^i, we get (3.2). Using (3.1), (2.1a) and (3.2) in (1.15), we get (3.3). Hence the infinitesimal transformation considered is an affine motion. Thus, we conclude

Theorem 4.1. *If a Finsler space admits an infinitesimal transformation generated by a concurrent vector , then the transformation is necessarily an affine motion.*

In view of (3.6) and (1.14), we get

(4.1) $$v^r\mathcal{B}_r U^i_{jkh} - U^r_{jkh}\mathcal{B}_r v^i + U^i_{rkh}\mathcal{B}_j v^r + U^i_{jrh}\mathcal{B}_k v^r + U^i_{jkr}\mathcal{B}_h v^r$$
$$+(\dot{\partial}_r U^i_{jkh})\mathcal{B}_s v^r y^s = 0.$$

Using (2.1b) in (4.1), we get

(4.2) $$v^m\mathcal{B}_m U^i_{jkh} + 2cU^i_{jkh} = 0.$$

Differentiating (4.2) covariantly with respect to x^l in the sense of Berwald and using (2.1b), we get

(4.3) $$v^m\mathcal{B}_l\mathcal{B}_m U^i_{jkh} + 3c\mathcal{B}_l U^i_{jkh} = 0.$$

Thus, we conclude

Theorem 4.2. *If a Finsler space admits an infinitesimal transformation generated by a concurrent vector, then the hv-curvature tensor U^i_{jkh} satisfies* (4.2) *and* (4.3).

If the space an $UR - F_n$, which is characterized by (1.8), is symmetric and denoted by $SUR - F_n$ .Thus, the $SUR - F_n$ is characterized by

(4.4) $\mathcal{B}_m \mathrm{U}^i_{jkh} = 0.$

In view of (4.2) and (4.4), we get $c\mathrm{U}^i_{jkh} = 0$, which implies $U^i_{jkh} = 0$ for $c \neq 0$.

Thus, we see that a symmetric recurrent space admitting an infinitesimal transformation generated by a concurrent vector is necessarily flat*.

Thus, we conclude

Corollary 4.1. *A non-flat $SUR - F_n$ does not admit any infinitesimal transformation generated by a concurrent vector.*

If the space $UBR - F_n$, which is characterized by (1.9), is bisymmetric and denoted by

$SUBR - F_n$. Thus, the $SUBR - F_n$ is characterized by

(4.5) $\mathcal{B}_l \mathcal{B}_m \mathrm{U}^i_{jkh} = 0.$

In view of (4.2), (4.3) and (4.5), we get $c\mathrm{U}^i_{jkh} = 0$, which implies $U^i_{jkh} = 0$ for $c \neq 0$. Thus, we see that a bisymmetric space admitting an infinitesimal transformation generated by a concurrent vector is necessarily flat.

Thus, we conclude

Corollary 4.2. *A non-flat $SUBR - F_n$ does not admit any infinitesimal transformation generated by a concurrent vector.*

5. Special Concircular Affine Motion

Let us consider an infinitesimal transformation generated by concurrent vector $v^i(x^j)$ characterized by (2.1c).

Hence, the infinitesimal transformation considered is an affine motion, we get $L_v \Pi^i_{jk} = 0$.

In view of (1.15), (1.19), (2.1c), (1.5) and (3.2), we get

(5.1) $\mathcal{B}_j \rho \delta^i_k = 0.$

Transvecting (5.1) by y^k and using the fact ($\mathcal{B}_j y^k = 0$), we get

(5.2) $y^i \mathcal{B}_j \rho = 0.$

Transvecting (5.2) by y_i and using the fact ($y^i y_i = F^2$), we get

(5.3) $F^2 \mathcal{B}_j \rho = 0$

which implies $\mathcal{B}_j\rho = 0$, a contradiction

Thus, we conclude

Theorem 5.1. *A Finsler space does not admit any special concircular affine motion*

A Finsler space with vanishing hv-curvature tensor is called *flat*.

6. Recurrent Affine Motion

Let us consider an infinitesimal transformation generated by concurrent vector $v^i(x^j)$ characterized by (2.1d)

Differentiating (2.1d) covariantly with respect to x^j in the sense of Berwald and using (2.1d), we get

(6.1) $\mathcal{B}_j\mathcal{B}_k v^i = (\mathcal{B}_j\mu_k+\mu_j\mu_k)v^i.$

In view of (1.15), (6.1), (2.1d) and putting $\mu = \mu_h y^h$, we get

(6.2) $L_v \Pi^i_{jk} = (\mathcal{B}_j\mu_k+\mu_j\mu_k)v^i - \mu U^i_{rjk}v^r + N^i_{rjk}v^r$

Differentiating (2.1d) partially with respect to y^j, we get

(6.3) $\dot{\partial}_j\mathcal{B}_k v^i = \dot{\partial}_j(\mu_k v^i)$

Transvecting (4.1) by y^j, using the fact ($\mathcal{B}_r y^i = 0$), (1.5), (2.1d) and putting $\mu = \mu_t y^t$, , we get

(6.4) $U^i_{rkh}v^r = 0,$

where $\mu \neq 0$.

Taking skew-symmetric part of (6.3), using the commutation formula exhibited by (1.6) for v^i and (6.4), we get

(6.5) $\dot{\partial}_j(\mu_k v^i) = 0.$

Transvecting (6.5) by y^k and putting $\mu = \mu_s y^s$, we get

(6.6) $\mu_j = \dot{\partial}_j\mu.$

Taking skew-symmetric part of (6.1) and using the commutation formula exhibited by (1.7) for v^i, we get

(6.7) $N^i_{rjk}v^r = (\mathcal{B}_j\mu_k+\mu_j\mu_k)v^i.$

In view of (1.19), (6.4) and (6.7), equ. (6.2) becomes

(6.8) $\mathcal{B}_j\mu_k + \mu_j\mu_k = 0.$

Using (6.7) in (6.8), we get

(6.9) $N^i_{rjk}v^r = 0.$

Thus, we see that the condition (6.8) is necessary consequence of a recurrent affine motion. Now, we shall establish that condition (6.8) is sufficient for (2.1d) to be an affine motion. To prove this, let us consider (6.8) holds.

Taking skew-symmetric part of (6.8), we get

(6.10) $\mathcal{B}_j\mu_k - \mathcal{B}_k\mu_j = 0.$

Transvecting (4.1) by y^j and using (1.5), we get

(6.11) $U^i_{rkh}y^j\mathcal{B}_jv^r = 0.$

In view of (2.1d) and putting $\mu = \mu_j y^j$, equ. (6.11) becomes

(6.12) $U^i_{rkh}v^r = 0,$

where $\mu \neq 0$.

In view of (6.8), (6.9) and (6.12), equ. (6.2) becomes

(6.13) $L_v\Pi^i_{jk} = 0.$

Hence, the transformation considered is an affine motion.

Thus, we conclude

Theorem 6.1. *The condition (6.8) is necessary and sufficient for an infinitesimal transformation generated by a recurrent vector $v^i(x^j)$ characterized by (2.1d) to be an affine motion.*

7. Torse Forming Affine Motion

Let us consider an infinitesimal transformation generated by concurrent vector $v^i(x^j)$ characterized by (2.1e).

Differentiating (2.1e) covariantly with respect to x^j in the sense of Berwald, we get

(7.1) $\mathcal{B}_j\mathcal{B}_kv^i = (\mathcal{B}_j\mu_k + \mu_j\mu_k)v^i + \rho\mu_k\delta^i_j + \rho_j\delta^i_k,$

where $\rho_j = \mathcal{B}_j \rho$.

Taking skew-symmetric part of (7.1) and using the commutation formula exhibited by (1.7) for v^i, we get

(7.2) $N^i_{rjk} v^r = (\mathcal{B}_j \mu_k + \mu_j \mu_k) v^i + \rho \mu_k \delta^i_j + \rho_j \delta^i_k$.

In view of (1.15), (1.19), (2.1e), putting $\mu = \mu_s y^s$, using (7.2) and (1.5), we get

(7.3) $-\mu U^i_{rjk} v^r + 2 N^i_{rjk} v^r = 0$.

Transvecting (4.1) by y^j and using (1.5), we get

(7.4) $U^i_{rkh} y^j \mathcal{B}_j v^r = 0$.

In view of (2.1e), putting $\mu = \mu_j y^j$ and (1.5) in (7.4), we get

(7.5) $\mu U^i_{rkh} v^r = 0$.

Using (7.5) in (7.3), we get

(7.6) $N^i_{rjk} v^r = 0$.

In view of (1.15), (7.1),(7.2), (7.5) and (7.6), we get

(7.7) $L_v \Pi^i_{jk} = 0$.

Hence, the transformation considered is an affine motion.

Thus, we conclude

Theorem 7.1. *The conditions (7.5) and (7.6) are necessary and sufficient for an infinitesimal transformation generated by a recurrent vector $v^i(x^j)$ characterized by (2.1e) to be an affine motion.*

8. Projective Motion of a Recurrent Finsler Space

The infinitesimal transformation (1.10) defines a projective motion if it transforms a system of geodesics of F_n into geodesics $\overline{F_n}$. A necessary and sufficient condition that the infinitesimal transformation (1.10) defines a projective motion [16] which characterized by condition (1.20).

For some homogeneous scalar function P of degree one in y^i. For the homogeneity of P_h, it satisfies

(8.1) $P_h y^h = P$.

Also, Lie-derivative of the hv-curvature tensor U^i_{jkh} in such projective motion may calculated by differentiating (1.20) partially with respect to y^j, using the commutation formula exhibited by (1.17) for Π^i_{jk} and in view of remark 1.1., we get

(8.2) $$L_v U^i_{jkh} = \delta^i_k P_{jh} + \delta^i_h P_{jk},$$

where

(8.3) $$P_{jl} = \dot{\partial}_j P_l$$

for the homogeneity of P_{jl}, it satisfies

(8.4) $$P_{jl} y^l = 0.$$

Now, the projective motion becomes an affine motion, the condition

(8.5) $$L_v \Pi^i_{jk} = 0$$

holds.

proof

Let us consider a Finsler space characterized by (8.5).

In view of (8.5) and (1.20), we get

(8.6) $$\delta^i_k P_h + \delta^i_h P_k = 0.$$

Contracting the indies i and k in (8.6), we get

(8.7) $$(n+1) P_h = 0$$

which implies

(8.8) $$P_h = 0.$$

Conversely, if (8.8) is true, the equ. (1.20) reduces to $L_v \Pi^i_{jk} = 0$. The condition (8.8) is the necessary and sufficient condition for the infinitesimal transformation (1.20), which defines a projective motion to be an affine motion.

Also, the projective motion becomes an affine motion, the condition

(8.9) $$L_v U^i_{jkh} = 0$$

holds.

proof

Let us consider a Finsler space characterized by (8.9).

In view of (8.9) and (8.2), we get

(8.10) $\delta^i_k P_{hj} + \delta^i_h P_{jk} = 0.$

Contracting the indies i and k in (8.10), we get

(8.11) $(n+1)P_{hj} = 0$

which implies

(8.12) $P_{hj} = 0.$

Conversely, if (8.12) is true, the equ. (8.2) reduces to $L_v U^i_{jkh} = 0.$ The condition (8.12) is the necessary and sufficient condition for the infinitesimal transformation (8.2), which defines a projective motion to be an affine motion.

Definition 8.1. A recurrent Finsler space characterized by (1.8) in which the infinitesimal transformation (1.19) defines a projective motion, is called *projective recurrent Finsler space* briefly denoted by $UR - P\bar{F}_n$.

Applying Lie- operator to (1.8) and using (8.2), we get

(8.13) $L_v \mathcal{B}_m U^i_{jkh} = (L_v \lambda_m) U^i_{jkh} + \lambda_m \left(\delta^i_k P_{hj} + \delta^i_h P_{jk}\right).$

Thus, we conclude

Theorem 8.1. *In an UR- $P\bar{F}_n$, which admits projective motion, the equ. (8.13) holds.*

In view of (8.12) and (8.13), we get

(8.14) $L_v \mathcal{B}_m U^i_{jkh} = (L_v \lambda_m) U^i_{jkh}.$

Thus, we conclude

Theorem 8.2. *In an UR- $P\bar{F}_n$, if the projective motion becomes an affine motion , the equ. (8.14) is necessarily true.*

In view of the commutation formula exhibited by (1.16) for the hv- curvature tensor U^i_{jkh}, (1.19) and (8.9), we get

(8.15) $L_v \mathcal{B}_m U^i_{jkh} = 0.$

Since the projective motion becomes an affine motion in $UR - P\bar{F}_n$, in view of (8.14) and (8.15), we get

(8.16) $L_v \lambda_m = 0$

since $\bar{F}_n$ is non- flat space.

Thus, we conclude

Theorem 8.3. *In an UR-* $P\bar{F}_n$*, if the projective motion becomes an affine motion , the recurrence vector field* λ_m *satisfies the identity (8.16).*

Differentiating (8.16) partially with respect to y^s, we get

(8.17) $\dot{\partial}_s L_v \lambda_m = 0.$

In view of the commutation formula exhibited by (1.17) for λ_m and (8.17), we get

(8.18) $L_v \dot{\partial}_s \lambda_m = 0$

since $\bar{F}_n$ is non- flat space.

Thus, we conclude

Theorem 8.4. *In an UR-* $P\bar{F}_n$*, if the projective motion becomes an affine motion , the recurrence vector field* λ_m *satisfies the identity (8.18).*

9. Special Projective Motion of a Recurrent Finsler Space

Let us consider a Finsler space admits a concurrent projective motion characterized by (2.1b).

Differentiating (2.1b) covariantly with respect to x^j in the sense of Berwald, taking skew-symmetric part of the obtained equation and using the commutation formula exhibited by (1.7) for v^i, we get

(9.1) $N^i_{hjk} v^h = 0.$

In view of (1.17) and (1.20), we get

(9.2) $\mathcal{B}_j \mathcal{B}_k v^i - U^i_{rjk} y^s \mathcal{B}_s v^r + N^i_{rjk} v^r = \delta^i_j P_k + \delta^i_k P_j.$

Using (2.1b) and (9.1) in (9.2), we get

(9.3) $-cU^i_{rkh}y^r = \delta^i_k P_h + \delta^i_h P_k.$

Differentiating (9.3) covariantly with respect to x^m and using the fact ($\mathcal{B}_m y^r = 0$), we get

(9.4) $-cy^r \mathcal{B}_m U^i_{rkh} = \delta^i_k \mathcal{B}_m P_h + \delta^i_h \mathcal{B}_m P_k.$

Using (1.8) and (9.3) in (9.4), we get

(9.5) $\delta^i_k(\mathcal{B}_m P_h - \lambda_m P_h) + \delta^i_h(\mathcal{B}_m P_k - \lambda_m P_k) = 0.$

Contracting the indies i and k in (9.5), we get

(9.6) $\mathcal{B}_m P_h = \lambda_m P_h.$

Thus, we conclude

Theorem 9.1. *In an UR-* $P\bar{F}_n$*, which admits projective motion, if the* vector *filed* $v^i(x^j)$ *spans concurrent field, then the scalar function P is recurrent.*

If we adopt the similar process for (2.1c), we get the following theorem

Theorem 9.2. *In an UR-* $P\bar{F}_n$*, which admits projective motion, if the* vector *filed* $v^i(x^j)$ *spans special concircular field, then the scalar function P is recurrent.*

Let us consider a Finsler space admits a special concircular projective motion characterized by (2.1c).

Differentiating (2.1c) covariantly with respect to x^j in the sense of Berwald, taking skew-symmetric part of the obtained equation and using the commutation formula exhibited by (1.7) for v^i, we get

(9.7) $N^i_{hjk}v^h = \rho_j \delta^i_k,$

where $\rho_j = \mathcal{B}_j \rho.$

In view of (1.15), (1.20) , (2.1c), (9.7) and (1.5), we get

(9.8) $2\rho_j \delta^i_k = \delta^i_j P_k + \delta^i_k P_j.$

Transvecting (9.8) by v^j, we get

(9.9) $2\rho_j v^j \delta^i_k = v^i P_k + \delta^i_k P_j v^j.$

Contracting the indies i and k in (9.9), we get

(9.10) $2n\rho_j v^j = (n+1)P_j v^j$.

Transvecting (9.9) by v^k, we get

(9.11) $\rho_j v^j = P_j v^j$.

In view of (9.10) and (9.11), we get

(9.12) $P_j v^j = 0 = \rho_j v^j$.

Using (9.12) in (9.9), we get

(9.13) $P_k = 0$.

In view of (9.8) and (9.13), we get

(9.14) $\rho_j \delta_k^i = 0$.

Transvecting (9.14) by y^k, y_i successively and using that fact ($y^i y_i = F^2$), we get

(9.15) $\rho_j = 0$

i. e. ρ is a covariant constant, a contradiction.

Thus, we conclude

Theorem 9.3. *A Finsler space does not admit any special concircular projective motion.*

Let us consider a Finsler space admits a recurrent projective motion characterized by (2.1d).

P. N. Pandy [7], proved that, if $v^i(x^j)$ are components of a non-null vector, then the equation

(9.16) $av^i + by^i = 0$

implies $a = b = 0$.

Differentiating (2.1d) covariantly with respect to x^j in the sense of Berwald, taking skew-symmetric part of the obtained equation, using the commutation formula exhibited by (1.7) for v^i and (2.1d), we get

and using (1.7), we get

(9.17) $N^i_{hjk}v^h = (\mathcal{B}_j\mu_k + \mu_j\mu_k)v^i.$

Transvecting (4.1) by y^j and using (1.5), we get

(9.18) $U^i_{rkh}y^j\mathcal{B}_jv^r = 0.$

In view of (2.1d) and putting $\mu = \mu_jy^j$ in (9.18), we get

(9.19) $\mu U^i_{rkh}v^r = 0.$

In view of (1.15), (2.1d), (9.17), (9.19) and (1.20), we get

(9.20) $2(\mathcal{B}_j\mu_k + \mu_j\mu_k)v^i = \delta^i_jP_k + \delta^i_kP_j.$

Transvecting (9.20) by y^k and y^j successively, using $\mu = \mu_ly^l$, the fact ($\mathcal{B}_ky^j = 0$) and (8.1), we get

(9.21) $\{y^j(\mathcal{B}_j\mu) + \mu^2\}v^i = Py^i.$

In view of (9.16) and (9.21), we get

(9.22) $a)\ y^j\mathcal{B}_j\mu = -\mu^2$ and $b)\ P = 0.$

Thus, we conclude

Theorem 9.4. *If a Finsler space admits a recurrent projective motion, the* vector *filed* $v^i(x^j)$ *satisfies (9.22a) and (9.22b).*

Transvecting (9.20) by v^j, we get

(9.23) $2(\mathcal{B}_j\mu_k + \mu_j\mu_k)v^iv^j = v^iP_k + \delta^i_kP_jv^j.$

In view of (1.12), (2.1d) and putting $\mu = \mu_sy^s$, Lie derivative of the vector μ_k is given by

(9.24) $L_v\mu_k = v^j\big(\mathcal{B}_j\mu_k + \mu_j\mu_k + \mu\dot{\partial}_j\mu_k\big).$

Differentiating (2.1d) partially with respect to y^j, we get

(9.25) $\dot{\partial}_j\mathcal{B}_kv^i = \dot{\partial}_j(\mu_kv^i).$

Taking skew-symmetric part of (9.25), using the commutation formula exhibited by (1.6) for v^i, (2.1d) and (9.19), we get

(9.26) $\dot{\partial}_j(\mu_k v^i) = 0.$

Transvecting (9.26) by y^k and putting $\mu = \mu_k y^k$, we get

(9.27) $\mu_j = \dot{\partial}_j \mu.$

Transvecting (9.24) by y^k, using (1.13a), (9.27) and putting $\mu = \mu_k y^k$, we get

(9.28) $L_v \mu = v^j(\mathcal{B}_j \mu + \mu\mu_j).$

In view of (9.28) and (1.13a), we get

(9.29) $\mathcal{B}_j \mu + \mu\mu_j = 0.$

Transvecting (9.23) by y^k , using the fact ($\mathcal{B}_j y^k = 0$), putting $\mu = \mu_k y^k$, (8.1) and (9.29), we get

(9.30) $P v^i + y^i P_j v^j = 0.$

In view of (9.16) and (9.30), we get

(9.31) $a)\ P = 0$ and $b)\ P_j v^j = 0.$

Thus, we conclude

Theorem 9.5. *The conditions (9.31a) and (9.31b) are necessary and sufficient for a recurrent projective motion to be an affine motion in a Finsler space.*

Transvecting (9.23) by y^k , putting $\mu = \mu_k y^k$, using (8.3) and (9.28), we get

(9.32) $(2L_v \mu - P)v^i = P_j v^j y^i.$

In view of (9.16) and (9.32), we get

(9.33) $a)\ 2L_v \mu = P$ and $b)\ P_j v^j = 0.$

Transvecting (9.20) by y^k, putting $\mu = \mu_k y^k$ and (8.1),we get

(9.34) $2(\mathcal{B}_j \mu + \mu\mu_j)v^i = \delta^i_j P + P_j y^i.$

Contracting the indies i and j in (9.34), using (9.29) and (8.1), we get

(9.35) $P = 0.$

Thus, we conclude

Theorem 9.6. *The condition (9.35) is necessary and sufficient for a recurrent projective motion to be an affine motion in a Finsler space.*

10. Conclusion

(10.1) Every contra vector generates an affine motion in a Finsler space.

(10.2) In an $UR - F_n$, if any contra vector $v^i(x^j)$ generates an infinitesimal transformation, it must be orthogonal to the recurrence vector.

(10.3) In an $UBR - F_n$, if any contra vector $v^i(x^j)$ generates an infinitesimal transformation, then the recurrence tensor a_{lm} satisfies conditions (3.9a) and (3.9b).

(10.4) If a Finsler space admits an infinitesimal transformation generated by a concurrent vector , then the transformation is necessarily an affine motion.

(10.5) If a Finsler space admits an infinitesimal transformation generated by a concurrent vector, then the hv-curvature tensor U^i_{jkh} satisfies (4.2) and (4.3).

(10.6) A Finsler space does not admit any special concircular affine motion.

(10.7) In an UR- $P\bar{F}_n$, if the projective motion becomes an affine motion.

(10.8) In an UR- $P\bar{F}_n$, if the projective motion becomes an affine motion , the recurrence vector field λ_m satisfies the identity (8.16).

(10.9) In an UR- $P\bar{F}_n$, if the projective motion becomes an affine motion , the recurrence vector field λ_m satisfies the identity (8.18).

(10.10) In an UR- $P\bar{F}_n$, which admits projective motion, if the vector filed $v^i(x^j)$ spans concurrent field, then the scalar function P is recurrent.

(10.11) A Finsler space does not admit any special concircular projective motion.

(10.12) If a Finsler space admits a recurrent projective motion, the vector filed $v^i(x^j)$ satisfies (9.22a) and (9.22b).

11. Recommendations

The authors recommend the research should be continued in the motions.

References

[1]. **Kumar, A.**" Some Theorems on Affine Motion in a Recurrent Finsler Space". IV, Indian J. Pure Appl. Math., 8, (1977), pp672-684.

[2]. **Kumar, A.**" On the Existence of Affine Motion in a Recurrent Finsler Space". Indian J. Pure Appl. Math., 8, (1977), pp791-800.

[3]. **Kumar, A.''** On Some Type of Affine Motion in Birecurrent Finsler Space". II, Indian J. Pure Appl. Math., 8, (1977), pp505-513.

[4]. **Kumar, A.**, **Shulka H. S.** and **Tripathi R. P''** On the Existence of Projective Affine Motion in a W-Recurrent Finsler Space".Tamkang J. of Math., Vol. 31, No. 1, Spring 2000.

[5]. **Meher, F. M.**" An SHR-F_n Admitting an Affine Motion II". Tensor N. S., 23, 1973,pp 208-210.

[6]. **Misra, R. B.**" A Turning Point in the Theory of Recurrent Finsler Manifolds".J. South Gujrat Univ., 6, (1977), pp72-96.

[7]. **Pandey, P.N.**" Some Problems in Finsler Spaces". D. Sc. Thesis, University of Allahabad, India, 1993.

[8]. **Pandey, P.N.** and **Dwivedi, V.J.**"Projective Motion in an NPR-Finsler Spaces". Tamkang J. Math., 17 (1), 1986,pp 87-98.

[9]. **Pandey, P.N.**, **Qasem, F.Y.A.** and **Suinta Pal''**Certain Types of Affine Motion".Proceeding of the third Conference of International Academy of Physical Sciences, The Indian Press Pvt. Ltd., Allahabad Indian, 2000, pp249-264.

[10]. **Qasem, F.Y.A.**" On Transformation in Finsler Spaces".D.Phil. Thesis, University of Allahabad, India, 2000.

[11]. **Qasem, F.Y.A.** and **Saleem, A.A.M.**: On U-Birecurrent Finsler Spaces, Univ. Aden J. Nat. and Appl. Sc., Vol. 14, 2010, pp587-596.

[12]. **Rund, H.** The differential geometry of Finsler spaces, Springer-verlag, Berlin Göttingen-Heidelberg, 1959; 2nd Edit. in Russian, Nauka, Moscow, 1981.

[13]. **Singh, S. P.**" On Projective Motion in Finsler Space". Prog. of Math., 36, 2002, pp151-158.

[14]. **Singh, S. P.**" Projective Motion in Birecurrent Finsler Space". Differential Geometry- Dyhamical Systems, Vol. 12, 2010,pp 221-227.

[15]. **Takano, K.** and **Imai, T.**" On Some Types of Affine Motion in Birecurrent Space'. Tensor N.S., 23, 1972, pp309-313.

[16]. **Yano, K.**"The Theory of Lie-Derivatives and its Applications". North – Holland publishing Co., Amsterdam, 1957, pp197-201.