International Journal of Sciences: Basic and Applied Research (IJSBAR)

ISSN 2307-4531
(Print & Online)

http://gssrr.org/index.php?journal=JournalOfBasicAndApplied

Some Eight - Step Implicit Linear Multistep Methods of Order Ten

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Abstract

In this paper we analyze the Taylor series method of deriving linear multistep methods through expansion of the linear difference operator $\mathcal{L}[y(x);h] = \sum_{j=0}^{k}[\alpha_j\, y(x+jh) - h\beta_j y'(x+jh)]$ where $y(x)$ is an arbitrary function continuously differentiable on $[a,b]$. The resulting constant expressions D_i $(i = 0(1)11)$ are expanded and solved accordingly. By a careful and judicious assignment of appropriate values to the free parameters, we obtain two eight – step implicit linear multistep schemes of optimal order (in this case order ten). The schemes are shown to be consistent and zero – stable; thereby establishing their convergence. In order to affirm their efficacy and reliability, the schemes are applied to sample initial value problems and the results compared to exact solutions. The negligibility of the exhibited errors further confirmed their usefulness.

Keywords: Linear multistep method; Consistency; Zero – stability; Convergence; Taylor series.

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1. Introduction

A linear multistep method of step number k (or a linear $k-$ step method) for solving the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{1}$$

has the general form

$$\sum_{\mathrm{j}=0}^{\mathrm{k}} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{2}$$

where $\alpha_k \neq 0$. Also $|\alpha_0| + |\beta_0| \neq 0$. More so, if $\beta_k = 0$ the method is explicit, if $\beta_k \neq 0$ it is called implicit [1]. A great many numerical methods for solving ordinary differential equations abound in literature, and yet many more are still being produced. One common feature of all improvements on such methods is the desire to increase the order of exactness of a numerical approximation. According to [2], a numerical method is r^{th} order exact if the expression in h^r in the Taylor expansion of the unknown is exactly replicated. This is determined by $O(h^r)$. The order r permits us to tell by how much the findings are enhanced when the step is lessened. Generally, methods with a big r are preferable since a diminution of h results in a large gain in accuracy. Obviously a high step linear multistep method (LMM) translates to high order, even though it comes at a cost; there is the twin problem of starting values and computational complexity often associated with such methods, which makes them less attractive. A number of authors have derived some high order LMMs in the past, these include [3,4,5,6]. This paper seeks to derive two eight – step implicit LMMs of order ten. Being implicit methods, a nine – step tenth - order Adams – Bashforth method is co-opted to serve as a predictor for the methods. In order to generate the starting values needed for the methods to kick off, a tenth order Runge – Kutta (RK) method used to do just that.

2. Materials and methods

2.1. Derivation of eight – step implicit linear multistep method

Since we seek to derive eight-step tenth-order implicit linear multistep, all the roots of the first characteristics polynomial $\rho(\xi)$ must be on the unit circle. Also, $\rho(\xi)$ is a polynomial of degree 8, and hence, by consistency, it has one real root at $+1$ and one more real root at -1. The remaining six roots must be complex. Consequently, the roots of $\rho(\xi)$ are computed as follows.

$$\left.\begin{array}{c} \alpha_0 = -1,\ \ \alpha_1 = 2(x+y+z), \qquad \alpha_2 = -2(1+2xy+2xz+2yz) \\ \alpha_3 = 2(x+y+z+4xyz), \qquad \alpha_4 = 0, \qquad \alpha_5 = -2(x+y+z+4xyz) \\ \alpha_6 = 2(1+2xy+2xz+2yz), \qquad \alpha_7 = -2(x+y+z), \qquad \alpha_8 = +1 \end{array}\right\} \tag{3}$$

And the order conditions for a tenth – order implicit linear multistep method are expressed in terms of the

$D_q,\ q = 0,1,2,\cdots,11$ as follows.

$$D_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 \tag{4}$$

$$D_1 = [-r\alpha_0 + (1-r)\alpha_1 + (2-r)\alpha_2 + \cdots + (7-r)\alpha_7 + (8-r)\alpha_8]$$

$$-[\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8] \tag{5}$$

$$D_2 = \frac{1}{2!}[(-r)^2\alpha_0 + (1-r)^2\alpha_1 + (2-r)^2\alpha_2 + \cdots + (7-r)^2\alpha_7 + (8-r)^2\alpha_8]$$

$$-[-r\beta_0 + (1-r)\beta_1 + (2-r)\beta_2 + \cdots + (7-r)\beta_7 + (8-r)\beta_8] \tag{6}$$

$$D_3 = \frac{1}{3!}[(-r)^3\alpha_0 + (1-r)^3\alpha_1 + (2-r)^3\alpha_2 + \cdots + (7-r)^3\alpha_7 + (8-r)^3\alpha_8]$$

$$-\frac{1}{2!}[-r^2\beta_0 + (1-r)^2\beta_1 + (2-r)^2\beta_2 + \cdots + (7-r)^2\beta_7 + (8-r)^2\beta_8] \tag{7}$$

$$D_4 = \frac{1}{4!}[(-r)^4\alpha_0 + (1-r)^4\alpha_1 + (2-r)^4\alpha_2 + \cdots + +(7-r)^4\alpha_7 + (8-r)^4\alpha_8]$$

$$-\frac{1}{3!}[-r^3\beta_0 + (1-r)^3\beta_1 + (2-r)^3\beta_2 + \cdots + +(7-r)^3\beta_7 + (8-r)^3\beta_8] \tag{8}$$

$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

$$D_9 = \frac{1}{9!}[(-r)^9\alpha_0 + (1-r)^9\alpha_1 + (2-r)^9\alpha_2 + \cdots + (7-r)^9\alpha_7 + (8-r)^9\alpha_8]$$

$$-\frac{1}{8!}[-r^8\beta_0 + (1-r)^8\beta_1 + (2-r)^8\beta_2 + \cdots + (7-r)^8\beta_7 + (8-r)^8\beta_8] \tag{9}$$

$$D_{10} = \frac{1}{10!}[(-r)^{10}\alpha_0 + (1-r)^{10}\alpha_1 + \cdots + (7-r)^{10}\alpha_7 + (8-r)^{10}\alpha_8]$$

$$-\frac{1}{9!}[-r^9\beta_0 + (1-r)^9\beta_1 + (2-r)^9\beta_2 + \cdots + (7-r)^9\beta_7 + (8-r)^9\beta_8] \tag{10}$$

$$D_{11} = \frac{1}{11!}[(-r)^{11}\alpha_0 + (1-r)^{11}\alpha_1 + \cdots + (7-r)^{11}\alpha_7 + (8-r)^{11}\alpha_8]$$

$$-\frac{1}{10!}[-r^{10}\beta_0 + (1-r)^{10}\beta_1 + (2-r)^{10}\beta_2 + \cdots + (7-r)^{10}\beta_7 + (8-r)^{10}\beta_8] \tag{11}$$

The free parameter r in Equations (5) to (11) for the $D_q = 0, q = 2,3,\cdots\cdots 10$, is chosen to be 4. On further simplification the resulting expressions for the $\beta_i, i = 0\ to\ 8$ are expressed thus.

$$\beta_0 = \frac{23}{14175}xyz + \frac{52}{14175}xy + \frac{52}{14175}xz + \frac{52}{14175}yz + \frac{188}{14175}x$$

$$+\frac{188}{14175}y + \frac{188}{14175}z + \frac{3982}{14175} = \beta_8 \qquad (12)$$

$$\beta_1 = -\frac{334}{14175}xyz - \frac{128}{2025}xy - \frac{128}{2025}xz - \frac{128}{2025}yz - \frac{9844}{14175}x$$

$$-\frac{9844}{14175}y + \frac{9844}{14175}z + \frac{23104}{14175} = \beta_7 \qquad (13)$$

$$\beta_2 = \frac{2804}{14175}xyz + \frac{21976}{14175}xy + \frac{21976}{14175}xz + \frac{21976}{14175}yz - \frac{37936}{14175}x$$

$$-\frac{37936}{14175}y - \frac{37936}{14175}z + \frac{7276}{14175} = \beta_6 \qquad (14)$$

$$\beta_3 = -\frac{46378}{14175}xyz + \frac{70528}{14175}xy + \frac{70528}{14175}xz + \frac{70528}{14175}yz - \frac{27268}{14175}x$$

$$-\frac{27268}{14175}y - \frac{27268}{14175}z + \frac{77248}{14175} = \beta_5 \qquad (15)$$

$$\beta_4 = -[\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_5 + \beta_6 + \beta_7 + \beta_8] + 8$$

$$12(x + y + z) + 8(1 + 2xy + 2xz + 2yz) + 4(x + y + z) \qquad (16)$$

The expression for the error constant D_{11} is therefore

$$D_{11} = \frac{2}{11!}[-4^{11}\alpha_0 - 3^{11}\alpha_1 - 2^{11}\alpha_2 - \alpha_3] - \frac{2}{10!}[4^{10}\beta_0 + 3^{10}\beta_1 + 2^{10}\beta_2 + \beta_3] \quad (17)$$

After assigning the values $-12/17$, $3/4$ and $2/3$ to the free parameters x, y and z respectively, in Equations (12) to (16) the following eight – step implicit linear multistep method of order ten is obtained.

$$y_{n+8} - \frac{145}{102}y_{n+7} + \frac{143}{102}y_{n+5} - \frac{143}{102}y_{n+3} + \frac{145}{102}y_{n+1} - y_n = h[\frac{208157}{722925}f^{(p)}{}_{n+8}$$

$$+\frac{850319}{722925}f_{n+7} - \frac{1614964}{722925}f_{n+6} + \frac{1987523}{722925}f_{n+5} - \frac{243554}{144585}f_{n+4} + \frac{1987523}{722925}f_{n+3}$$

$$-\frac{1614964}{722925}f_{n+2} + \frac{850319}{722925}f_{n+1} + \frac{208157}{722925}f_n] \qquad (18)$$

and the corresponding error constant is calculated from Equation (17) as $-\frac{424814429}{72142131600}$.

Similarly, substituting $x = \frac{1}{2}$, $y = 0$, $z = -\frac{1}{2}$ in Equations (12 to (16) the values of the $\alpha_i's$ and $\beta_i's$ are obtained for the resultant eight - step implicit linear multistep method of order ten

$$y_{n+8} + y_{n+6} - y_{n+2} - y_n = h[\frac{7}{25} f^{(p)}{}_{n+8} + \frac{288}{175} f_{n+7} + \frac{22}{175} f_{n+6} + \frac{736}{175} f_{n+5} -$$

$$\frac{18}{35} f_{n+4} + \frac{736}{175} f_{n+3} + \frac{22}{175} f_{n+2} + \frac{288}{175} f_{n+1} + \frac{7}{25} f_n] \quad (19)$$

whose error constant is calculated to be $-\frac{2}{385}$.

2.2. Convergence analysis

For a linear multistep method, consistency demands that

$$\left.\begin{array}{l}(i) \quad \rho(1) = 0 \\ (ii) \;\; \rho'(1) = \sigma(1)\end{array}\right\} \quad (20)$$

Thus for the LMM (18)

$$\rho(\xi) = \xi^8 - \frac{145}{102}\xi^7 + \frac{143}{102}\xi^5 - \frac{143}{102}\xi^3 + \frac{145}{102}\xi - 1 \quad (21)$$

$$\rho(1) = 1 - \frac{145}{102} + \frac{143}{102} - \frac{145}{102} + \frac{145}{102} - 1 = 0 \quad (22)$$

$$\rho'(1) = 8(1) - \frac{1015}{102}(1) + \frac{715}{102}(1) - \frac{429}{102}(1) + \frac{145}{102}(1) = \frac{116}{51} \quad (23)$$

$$\sigma(\xi) = \sum_{j=0}^{8} \beta_j \xi^j = \frac{208157}{722925}\xi^8 + \frac{850319}{722925}\xi^7 - \frac{1614964}{722925}\xi^6 + \frac{1987523}{722925}\xi^5 - \frac{243554}{144585}\xi^4$$

$$+ \frac{1987523}{722925}\xi^3 - \frac{1614964}{722925}\xi^2 + \frac{850319}{722915}\xi + \frac{208157}{722925} \quad (24)$$

$$\sigma(1) = \frac{116}{51} \quad (25)$$

And from Equations (22), (23) and (25) the the consistency of LMM (18) is established.

Zero –stability entails that no root of the first characteristic polynomial, Equation (21), has modulus greater than 1, and every root with modulus 1 is simple. The roots of Equation (21) are computed to be

$$\left.\begin{array}{c}\xi_1 = 1, \xi_2 = -1, \xi_3 = \frac{3}{4} - \frac{1}{4}\sqrt{7}\, i, \xi_4 = \frac{3}{4} + \frac{1}{4}\sqrt{7}\, i, \xi_5 = -\frac{12}{17} - \frac{1}{17}\sqrt{145}\, i \\ \xi_6 = -\frac{12}{17} + \frac{1}{17}\sqrt{145}\, i, \quad \xi_7 = \frac{2}{3} - \frac{1}{3}\sqrt{5}\, i, \quad \xi_8 = \frac{2}{3} + \frac{1}{3}\sqrt{5}\, i\end{array}\right\} \quad (26)$$

All the roots $\xi_i, i = 1,2, \cdots 8$, of Equations (26) are calculated to have a modulus of 1 each, thereby establishing the zero – stability of the scheme (18).

In the same vein, for the LMM (18)

$$\rho(\xi) = \rho(\xi) = \xi^8 + \xi^6 - \xi^2 - 1 \qquad (27)$$

$$\rho(1) = 1 + 1 - 1 - 1 = 0 \qquad (28)$$

$$\rho'(1) = 8(1) + 6(1) - 2(1) = 12 \qquad (29)$$

$$\sigma(\xi) = \sum_{j=0}^{8} \beta_j \xi^j = \frac{7}{25}\xi^8 + \frac{288}{175}\xi^7 - \frac{22}{175}\xi^6 + \frac{736}{175}\xi^5 - \frac{18}{35}\xi^4 + \frac{7}{25}\xi^3 - \frac{288}{175}\xi^2$$

$$+ \frac{22}{175}\xi + \frac{736}{175} \qquad (30)$$

$$\sigma(1) = 12 \qquad (31)$$

And from Equations (28), (29) and (31) the the consistency of LMM (19) is established.

Zero –stability entails that no root of the first characteristic polynomial, Equation (27), has modulus greater than 1, and every root with modulus 1 is simple. The roots Equation (27) are computed to be

$$\left.\begin{array}{c}\xi_1 = 1, \xi_2 = -1, \xi_3 = i, \xi_4 = -i, \xi_5 = \frac{1}{2}\sqrt{-2 - 2\sqrt{3}\, i} \\ \xi_6 = -\frac{1}{2}\sqrt{-2 - 2\sqrt{3}\, i}, \quad \xi_7 = \frac{1}{2}\sqrt{-2 + 2\sqrt{3}\, i}, \quad \xi_8 = -\frac{1}{2}\sqrt{-2 + 2\sqrt{3}\, i}\end{array}\right\} (32)$$

All the roots $\xi_i, i = 1,2, \cdots 8$, of Equation (32) are calculated to have a modulus of 1 each, thereby establishing the zero – stability of the scheme (19).

Therefore, it is established that the eight – step implicit linear multistep methods (18) and (19) are convergent.

2.3. Numerical experiments

Two sample problems are solved with the derived methods (18) and (19) to further demonstrate their efficiency and effectiveness.

Problem 1: $y' = 7x^6 - 10x^4 + 9x^2 + 2, \; y(0) = 1, \; h = 0.1, \; 0 \le x \le 2$

Exact solution: $y_E(x) = x^7 - 2x^5 + 3x^3 + 2x + 1$

Problem 2: $y' = -y, \quad y(0) = 1, \; h = 0.1, \; 0 \le x \le 2$

Exact solution: $y_E(x) = e^{-x}$

In order to generate the necessary starting values for the eight – step LMMs of order ten, a tenth – order Runge – Kutta method introduced by Hairer [7] is used, while a nine – step Adams – Bashforth method of order ten due to [6] is provided to serve as a predictor to the implicit LMMs thus.

$$y_{n+1} = y_n + \frac{h}{7257600}[49537553f_n - 259077637f_{n-1} + 805221248f_{n-2}$$

$$-1533238912f_{n-3} + 1886585258f_{n-4} - 1523349298f_{n-5} + 791906792f_{n-6}$$

$$-248389768f_{n-7} + 401445117f_{n-8} - 2082753f_{n-9}] \qquad (33)$$

The computations are done using Maple software package and the results presented in Tables 1 to 4.

3. Results

A **Table 1:** Results of Problem 1 with Scheme (18)

x	**Exact solution**	**Approximate**	**Error**
0.0	1.0000000000	1.0000000000	0.0000000000E+00
0.1	1.2029801000	1.2029801000	0.0000000000E+00
0.2	1.4233728000	1.4233728000	0.0000000000E+00
0.3	1.6763587000	1.6763587000	0.0000000000E+00
0.4	1.9731584000	1.9731584000	0.0000000000E+00
0.5	2.3203125000	2.3203125000	0.0000000000E+00
0.6	2.7204736000	2.7204736000	0.0000000000E+00
0.7	3.1752143000	3.1752143000	0.0000000000E+00
0.8	3.6903552000	3.6903552000	0.0000000000E+00
0.9	4.2843169000	4.2843169000	0.0000000000E+00
1.0	5.0000000000	5.0000000000	0.0000000000E+00
1.1	5.9206971000	5.9206971000	0.0000000000E+00
1.2	7.1905408000	7.1905408000	0.0000000000E+00
1.3	9.0399917000	9.0399917000	0.0000000000E+00
1.4	11.8168704000	11.8168704000	0.0000000000E+00
1.5	16.0234375000	16.0234375000	0.0000000000E+00

1.6	22.3600256000	22.3600256000	0.0000000000E+00
1.7	31.7757273000	31.7757273000	0.0000000000E+00
1.8	45.5266432000	45.5266432000	0.0000000000E+00
1.9	65.2421939000	65.2421939000	0.0000000000E+00
2.0	93.0000000000	93.0000000000	0.0000000000E+00

The results of Tables 1 and 2 revealed that the two LMMs solved the differential equation exactly; this is as expected, since the exact solution of Problem 1 is a polynomial of degree 6, which is less than the step number of the LMMs, i.e., 8.

In Tables 3 and 4 the two LMMs exhibited high levels of accuracy, by solving the differential equation with very minimal errors as revealed in the solutions.

Table 2: Results of Problem 1 with Scheme (19)

x	**Exact solution**	**Approximate**	**Error**
0.0	1.0000000000	1.0000000000	0.0000000000E+00
0.1	1.2029801000	1.2029801000	0.0000000000E+00
0.2	1.4233728000	1.4233728000	0.0000000000E+00
0.3	1.6763587000	1.6763587000	0.0000000000E+00
0.4	1.9731584000	1.9731584000	0.0000000000E+00
0.5	2.3203125000	2.3203125000	0.0000000000E+00
0.6	2.7204736000	2.7204736000	0.0000000000E+00
0.7	3.1752143000	3.1752143000	0.0000000000E+00
0.8	3.6903552000	3.6903552000	0.0000000000E+00
0.9	4.2843169000	4.2843169000	0.0000000000E+00
1.0	5.0000000000	5.0000000000	0.0000000000E+00
1.1	5.9206971000	5.9206971000	0.0000000000E+00
1.2	7.1905408000	7.1905408000	0.0000000000E+00
1.3	9.0399917000	9.0399917000	0.0000000000E+00
1.4	11.8168704000	11.8168704000	0.0000000000E+00
1.5	16.0234375000	16.0234375000	0.0000000000E+00
1.6	22.3600256000	22.3600256000	0.0000000000E+00
1.7	31.7757273000	31.7757273000	0.0000000000E+00
1.8	45.5266432000	45.5266432000	0.0000000000E+00
1.9	65.2421939000	65.2421939000	0.0000000000E+00
2.0	93.0000000000	93.0000000000	0.0000000000E+00

Table 3: Results of Problem 2 with Scheme (18)

x	Exact solution	Approximate	Error
0.0	1.0000000000	1.0000000000	0.0000000000E+00
0.1	0.9048374180	0.9048374180	3.6010083804E-11
0.2	0.8187307531	0.8187307531	2.1927015759E-11
0.3	0.7408182207	0.7408182207	1.8158030635E-11
0.4	0.6703200460	0.6703200460	3.5789038400E-11
0.5	0.6065306597	0.6065306597	1.2801981697E-11
0.6	0.5488116361	0.5488116361	5.7900351180E-12
0.7	0.4965853038	0.4965853038	8.3970053133E-12
0.8	0.4493289641	0.4493289641	1.7422008280E-11
0.9	0.4065696597	0.4065696597	4.0803027623E-11
1.0	0.3678794412	0.3678794412	0.0000000000E+00
1.1	0.3328710837	0.3328710838	1.0000000827E-10
1.2	0.3011942119	0.3011942121	2.0000001655E-10
1.3	0.2725317930	0.2725317932	1.9999996104E-10
1.4	0.2465969639	0.2465969641	2.0000001655E-10
1.5	0.2231301601	0.2231301602	1.0000000827E-10
1.6	0.2018965180	0.2018965179	1.0000000827E-10
1.7	0.1826835241	0.1826835239	1.9999998879E-10
1.8	0.1652988882	0.1652988882	0.0000000000E+00
1.9	0.1495686192	0.1495686193	1.0000000827E-10
2.0	0.1353352832	0.1353352835	2.9999999707E-10

Table 4: Results of Problem 2 with Scheme (19)

x	Exact solution	Approximate	Error
0.0	1.0000000000	1.0000000000	0.0000000000E+00
0.1	0.9048374180	0.9048374180	3.6010083804E-11
0.2	0.8187307531	0.8187307531	2.1927015759E-11
0.3	0.7408182207	0.7408182207	1.8158030635E-11
0.4	0.6703200460	0.6703200460	3.5789038400E-11
0.5	0.6065306597	0.6065306597	1.2801981697E-11
0.6	0.5488116361	0.5488116361	5.7900351180E-12
0.7	0.4965853038	0.4965853038	8.3970053133E-12
0.8	0.4493289641	0.4493289641	1.7422008280E-11
0.9	0.4065696597	0.4065696597	4.0803027623E-11
1.0	0.3678794412	0.3678794412	0.0000000000E+00
1.1	0.3328710837	0.3328710837	0.0000000000E+00

1.2	0.3011942119	0.3011942119	0.0000000000E+00
1.3	0.2725317930	0.2725317930	0.0000000000E+00
1.4	0.2465969639	0.2465969640	1.0000000827E-10
1.5	0.2231301601	0.2231301602	1.0000000827E-10
1.6	0.2018965180	0.2018965180	0.0000000000E+00
1.7	0.1826835241	0.1826835241	0.0000000000E+00
1.8	0.1652988882	0.1652988882	0.0000000000E+00
1.9	0.1495686192	0.1495686193	1.0000000827E-10
2.0	0.1353352832	0.1353352832	0.0000000000E+00

4. Conclusion

By the foregoing, it is instructive that the two eight – step implicit linear multistep methods of order ten are convergent, effective and efficient in the solution of initial value problems.

5. Recommendation

The authors recommend further research be carried out towards development of low step methods that achieve high order accuracy in order to reduce the computational rigours involved in generating the starting values needed for high step methods.

Acknowledgements

The authors wish express their profound gratitude and appreciation to all those whose works were cited herein for their valuable contributions.

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