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Direct Product of Left Almost Rings

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Abstract

In this paper, we define direct product of left almost rings and show that it becomes a left almost ring. Further we characterize left almost rings by the properties of the direct product.

Keywords: Ideals; Isomorphisms; Direct Products.

1. Introduction and Preliminaries

In 1972, Kazim and Naseeruddin [3] introduced braces on the left of the equation $abc = cba$, and get a new pseudo associative law, that is $(ab)c = (cb)a$. It is known as left invertive law. A groupoid is called a left almost semigroup, abbreviated as LA-semigroup, if it satisfies the left invertive law. It corresponds to a semigroup and is basically the generalization of a commutative semigroup. In [6], LA-semigroup is also known as an Abel-Grassmann's groupoid (AG-groupoid) after the name of Abel-Grassmann.

In 1993, Kamran [5] extended the concept of LA-semigroups to left almost groups, abbreviated as LA-groups. An LA-group corresponds to a group. It is a non-associative structure and the generalization of commutative groups.

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Let ($\boldsymbol{S}$, $*$) be a groupoid, then it is called a left almost group if it satisfies the following conditions:

- Elements of $\boldsymbol{S}$ must satisfy the left invertive law. That is, $a(bc) = (cb)a$ for all $a, b, c \in \boldsymbol{S}$.
- There exists an element $e \in \boldsymbol{S}$ such that $e * s = s$ for all $s \in \boldsymbol{S}$. That is, left identity element exists in $\boldsymbol{S}$.
- For all $s \in \boldsymbol{S}$ there exists $s^{-1} \in \boldsymbol{S} \ni s * s^{-1} = s^{-1} * s = e$. That is, left inverse of each element of $\boldsymbol{S}$ exists in $\boldsymbol{S}$.

In [5], the author proved some interesting and elegant results about LA-groups. Particularly the author discussed substructures of LA-groups and then quotient structures.

In 2006, Yusuf [7] extended the concept of an LA-group to a non-associative structure called left almost ring, abbreviated as LA-ring. LA-rings basically correspond to rings. A left almost ring is a set $\boldsymbol{R} \neq \varphi$ with the binary operations '+' and '·' which satisfies the conditions below:

- ($\boldsymbol{R}$, +) is an LA-group,
- ($\boldsymbol{R}$, ·) is an LA-semigroup,
- Distributive laws of multiplication over addition hold in $\boldsymbol{R}$, i.e. $\forall\ a, b, c \in \boldsymbol{R}$;

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (b + c) \cdot a = b \cdot a + c \cdot a.$$

Further different peoples in [1], [2], [4] and [8] worked on LA-rings and explored many interesting and useful properties of LA-rings. In this paper, we study direct product of two left almost rings and explore some elegant properties.

2. Direct Products

In this section, we define direct products of two LA-rings and show that the said direct product becomes an LA-ring. We also discuss ideals of the direct product of two LA-rings. We then explore some properties of the direct product of two LA-rings which are based on isomorphism.

Definition 2.1

Let ($\boldsymbol{R}$, $+'$, $\cdot'$) and ($\boldsymbol{S}$, $+''$, $\cdot''$) be LA-rings. Then we can define addition and multiplication on the set $\boldsymbol{R} \times \boldsymbol{S} = \{(r, s) : r \in \boldsymbol{R} \text{ and } s \in \boldsymbol{S}\}$ as follows:

Let $(r_1, s_1), (r_2, s_2) \in \boldsymbol{R} \times \boldsymbol{S}$, then we define

$(r_1, s_1) + (r_2, s_2) = (r_1 +' r_2, s_1 +'' s_2)$ and $(r_1, s_1) \cdot (r_2, s_2) = (r_1 \cdot' r_2, s_1 \cdot'' s_2)$.

$\boldsymbol{R} \times \boldsymbol{S}$ with the above binary operations is called direct product of $\boldsymbol{R}$ and $\boldsymbol{S}$.

We now state and prove some properties of the direct products of two LA-rings $\boldsymbol{R}$ and $\boldsymbol{S}$. The following result shows that the direct product of two LA-rings $\boldsymbol{R}$ and $\boldsymbol{S}$ becomes an LA-ring.

Theorem 2.2

Let $(\boldsymbol{R}, +', \cdot')$ and $(\boldsymbol{S}, +'', \cdot'')$ be LA-rings. Then the direct product of $\boldsymbol{R}$ and $\boldsymbol{S}$ is an LA-ring under the above defined binary operations.

Proof.

- Closure property with respect to '+'

It is clear from the definition.

- Left invertive property with respect to '+'

Let (r_1, s_1), (r_2, s_2) and $(r_3, s_3) \in \boldsymbol{R} \times \boldsymbol{S}$, then

$$\begin{aligned}((r_1, s_1) + (r_2, s_2)) + (r_3, s_3) &= (r_1 +' r_2, s_1 +'' s_2) + (r_3, s_3)\\ &= ((r_1 +' r_2) +' r_3, (s_1 +'' s_2) +'' s_3)\\ &= ((r_3 +' r_2) +' r_1, (s_3 +'' s_2) +'' s_1)\\ &= (r_3 +' r_2, s_3 +'' s_2) + (r_1, s_1)\\ &= ((r_3, s_3) + (r_2, s_2)) + (r_1, s_1)\end{aligned}$$

- Left additive identity

As $0_{\boldsymbol{R}} \in \boldsymbol{R}$ and $0_{\boldsymbol{S}} \in \boldsymbol{S} \Rightarrow (0_{\boldsymbol{R}}, 0_{\boldsymbol{S}}) \in \boldsymbol{R} \times \boldsymbol{S}$.

Now let $(r, s) \in \boldsymbol{R} \times \boldsymbol{S}$, then

$$\begin{aligned}(0_{\boldsymbol{R}}, 0_{\boldsymbol{S}}) + (r, s) &= (0_{\boldsymbol{R}} +' r, 0_{\boldsymbol{S}} +'' s)\\ &= (r, s).\end{aligned}$$

Thus, it follows that $(0_{\boldsymbol{R}}, 0_{\boldsymbol{S}})$ is the left additive identity in $\boldsymbol{R} \times \boldsymbol{S}$.

- Left additive inverse

Let $(r, s) \in \boldsymbol{R} \times \boldsymbol{S} \Rightarrow r \in \boldsymbol{R}$ and $s \in \boldsymbol{S} \Rightarrow -'r \in \boldsymbol{R}$ and $-''s \in \boldsymbol{S} \Rightarrow (-'r, -''s) \in \boldsymbol{R} \times \boldsymbol{S}$.

Now

$(r, s) + (-'r, -''s) = (r +' (-'r), s +'' (-''s))$

$= (r -'r, s -''s)$

$= (0_{\boldsymbol{R}}, 0_{\boldsymbol{S}})$.

Similarly

$(-'r, -''s) + (r, s) = (0_{\boldsymbol{R}}, 0_{\boldsymbol{S}})$.

Thus, $(-'r, -''s)$ is left additive inverse of (r, s).

Therefore, $\boldsymbol{R} \times \boldsymbol{S}$ is a left almost group.

Now to show $\boldsymbol{R} \times \boldsymbol{S}$ is a left almost semigroup, we have

- Closure property with respect to '·'

It is clear from the definition.

- Left invertive law with respect to '·'

Let (r_1, s_1), (r_2, s_2) and $(r_3, s_3) \in \boldsymbol{R} \times \boldsymbol{S}$, then

$((r_1, s_1) \cdot (r_2, s_2)) \cdot (r_3, s_3) = (r_1 \cdot' r_2, s_1 \cdot'' s_2) \cdot (r_3, s_3)$

$= ((r_1 \cdot' r_2) \cdot' r_3, (s_1 \cdot'' s_2) \cdot'' s_3)$

$= ((r_3 \cdot' r_2) \cdot' r_1, (s_3 \cdot'' s_2) \cdot'' s_1)$

$= (r_3 \cdot' r_2, s_3 \cdot'' s_2) \cdot (r_1, s_1)$

$= ((r_3, s_3) \cdot (r_2, s_2)) \cdot (r_1, s_1)$.

Thus, $\boldsymbol{R} \times \boldsymbol{S}$ is an LA-semigroup. Further we prove

- Distributive laws

Let (r_1, s_1), (r_2, s_2) and $(r_3, s_3) \in \boldsymbol{R} \times \boldsymbol{S}$, then we have

$(r_1, s_1) \cdot ((r_2, s_2) + (r_3, s_3)) = (r_1, s_1) \cdot (r_2 +' r_3, s_2 +'' s_3)$

$= (r_1 \cdot' (r_2 +' r_3), s_1 \cdot'' (s_2 +'' s_3))$

$$= ((r_1 \cdot' r_2) +' (r_1 \cdot' r_3), (s_1 \cdot'' s_2) +'' (s_1 \cdot'' s_3))$$

$$= (r_1 \cdot' r_2, s_1 \cdot'' s_2) + (r_1 \cdot' r_3, s_1 \cdot'' s_3)$$

$$= ((r_1, s_1) \cdot (r_2, s_2)) + ((r_1, s_1) \cdot (r_3, s_3)).$$

It follows that $\boldsymbol{R} \times \boldsymbol{S}$ satisfies left distributive law. In the same way we may prove right distributive law. This completes the proof.

We are now going to state and prove a result in which we discuss ideals of the direct product of two LA-rings $\boldsymbol{R}$ and $\boldsymbol{S}$. Before stating it we are going to define left almost subring and ideal in left almost rings. The following definition has been taken from [7].

Definition 2.3

Let $\boldsymbol{S} \neq \varphi$ be a subset of the LA-ring $\boldsymbol{R}$. Then, $\boldsymbol{S}$ is known as a left almost subring (abbreviated as LA-subring) of $\boldsymbol{R}$ if $\boldsymbol{S}$ is itself a left almost ring under the same binary operations of $\boldsymbol{R}$.

Let us present some properties which have been taken from [7]. The following result gives us equivalent conditions for left almost subrings.

Theorem 2.4

Let $(\boldsymbol{R}, +, \cdot)$ be an LA-ring. Let $\varphi \neq \boldsymbol{S} \subseteq \boldsymbol{R}$, then $\boldsymbol{S}$ is a left almost subring of $\boldsymbol{R}$ if and only if $a - b \in \boldsymbol{S}$ and $a \cdot b \in \boldsymbol{S}$ $\forall\, a, b \in \boldsymbol{S}$.

We are now going to define ideals. The following definition has been taken from [7].

Definition 2.5

Let $(\boldsymbol{R}, +, \cdot)$ be an LA-ring and $\boldsymbol{I}$ an LA-subring of $\boldsymbol{R}$. If $\boldsymbol{RI} \subseteq \boldsymbol{I}$, then $\boldsymbol{I}$ is said to be a left ideal of $\boldsymbol{R}$ and if $\boldsymbol{IR} \subseteq \boldsymbol{I}$, then $\boldsymbol{I}$ is said to be a right ideal of $\boldsymbol{R}$. If $\boldsymbol{I}$ is both left and right ideal of $\boldsymbol{R}$, then $\boldsymbol{I}$ is said to be two sided ideal or an ideal of $\boldsymbol{R}$.

From the definition, it is clear that a non-empty subset $\boldsymbol{I}$ of an LA-ring $\boldsymbol{R}$ is said to be a left ideal if $a - b \in \boldsymbol{I}$ and $r \cdot a \in \boldsymbol{I}$ $\forall\, a, b \in \boldsymbol{I}$ and $r \in \boldsymbol{R}$. Similarly a non-empty subset $\boldsymbol{I}$ of an LA-ring is said to be a right ideal if $a - b \in \boldsymbol{I}$ and $a \cdot r \in \boldsymbol{I}$ $\forall\, a, b \in \boldsymbol{I}$ and $r \in \boldsymbol{R}$.

Now if $\boldsymbol{S}$ is an LA-subring of an LA-ring $(\boldsymbol{R}, +, \cdot)$. Then for all $x, y \in \boldsymbol{R}$, we write $x \equiv y \pmod{\boldsymbol{S}}$ if and only if $x - y \in \boldsymbol{S}$. The relation $\equiv$ becomes an equivalence relation in the LA-ring $\boldsymbol{R}$.

Let $x \in \boldsymbol{R}$ and $\boldsymbol{S} + x$ denotes the equivalence classes corresponding to x. Then we define

$\boldsymbol{R/S} = \{\boldsymbol{S} + x : x \in \boldsymbol{R}\}$.

Let $\boldsymbol{I}$ be an ideal of $\boldsymbol{R}$, then

$\boldsymbol{R/I} = \{\boldsymbol{I} + x : x \in \boldsymbol{R}\}$.

The following result [7] shows that, if $\boldsymbol{I}$ is an ideal of an LA-ring $\boldsymbol{R}$, then $\boldsymbol{R/I}$ become an LA-ring.

Theorem 2.6

Let $(\boldsymbol{R}, +, \cdot)$ be an LA-ring and $\boldsymbol{I}$ an ideal of $\boldsymbol{R}$. Then $\boldsymbol{R/I}$ is an LA-ring under the following binary operations:

$$(\boldsymbol{I} + s) + (\boldsymbol{I} + t) = \boldsymbol{I} + (s + t)$$

$$(\boldsymbol{I} + s) \cdot (\boldsymbol{I} + t) = \boldsymbol{I} + s \cdot t.$$

Let us present some properties of ideals which have been taken from the source [2].

Theorem 2.7

Let $\boldsymbol{I}$ and $\boldsymbol{J}$ be two left (right) ideals of an LA-ring $\boldsymbol{R}$. Then $\boldsymbol{I} \cap \boldsymbol{J}$ is also a left (right) ideal of $\boldsymbol{R}$.

It follows from the above theorem that the intersection of any family of left (right) ideals of an LA-ring is a left (right) ideal.

Theorem 2.8

Let $\boldsymbol{I}$ be a two sided ideal in $\boldsymbol{R}$ and $\boldsymbol{J}$ be a two sided ideal in $\boldsymbol{S}$. Then $\boldsymbol{I} \times \boldsymbol{J}$ is a two sided ideal in $\boldsymbol{R} \times \boldsymbol{S}$.

Proof.

First note that $\boldsymbol{I} \times \boldsymbol{J}$ is non-empty.

As $0_{\boldsymbol{R}} \in \boldsymbol{I}$ and $0_{\boldsymbol{S}} \in \boldsymbol{J} \Rightarrow (0_{\boldsymbol{R}}, 0_{\boldsymbol{S}}) \in \boldsymbol{I} \times \boldsymbol{J} \Rightarrow \boldsymbol{I} \times \boldsymbol{J}$ is non-empty.

Now let $(r_1, s_1) \in \boldsymbol{I} \times \boldsymbol{J}$ and $(r_2, s_2) \in \boldsymbol{I} \times \boldsymbol{J} \Rightarrow r_2 \in \boldsymbol{I}$ and $s_2 \in \boldsymbol{J} \Rightarrow -'r_2 \in \boldsymbol{I}$ and $-''s_2 \in \boldsymbol{J} \Rightarrow (-'r_2, -''s_2) \in \boldsymbol{I} \times \boldsymbol{J}$.

Now

$$(r_1, s_1) - (r_2, s_2) = (r_1, s_1) + (-'r_2, -''s_2)$$

$$= (r_1 +' (-'r_2), s_1 +'' (-''s_2))$$

$$= (r_1 -' r_2, s_1 -''s_2) \in \boldsymbol{I} \times \boldsymbol{J} \qquad \because r_1 -'r_2 \in \boldsymbol{I} \text{ and } s_1 -''s_2 \in \boldsymbol{J}.$$

Now let $(a, b) \in \boldsymbol{R} \times \boldsymbol{S}$ and $(r, s) \in \boldsymbol{I} \times \boldsymbol{J}$, then

$(a, b) \cdot (r, s) = (a \cdot' r, b \cdot'' s) \in \boldsymbol{I} \times \boldsymbol{J}$ $\because a \cdot' r \in \boldsymbol{I}$ and $b \cdot'' s \in \boldsymbol{J}$.

It follows that $\boldsymbol{I} \times \boldsymbol{J}$ is a left ideal. Similarly we can show that $\boldsymbol{I} \times \boldsymbol{J}$ is a right ideal.

We are now going to state and prove some results of the direct product of two LA-rings $\boldsymbol{R}$ and $\boldsymbol{S}$ which are based on isomorphisms. Before stating and proving them firstly we are going to define isomorphism of two LA-rings $\boldsymbol{R}$ and $\boldsymbol{S}$. The following definition has been taken from [7].

Definition 2.9

Let $(\boldsymbol{R}, +, \cdot)$ and $(\boldsymbol{S}, \oplus, \odot)$ be a left almost rings. Let $\alpha : \boldsymbol{R} \to \boldsymbol{S}$ be a mapping from $\boldsymbol{R}$ to $\boldsymbol{S}$, then α is said to be a homomorphism if:

- $(a + b)\alpha = (a)\alpha \oplus (b)\alpha$
- $(a \cdot b)\alpha = (a)\alpha \odot (b)\alpha$

for all $a, b \in \boldsymbol{R}$.

It should be noted that if $\boldsymbol{R} = \boldsymbol{S}$, then the homomorphism α is known as an endomorphism. If α is onto and a homomorphism, then α is known as an epimorphism. If α is one-one and a homomorphism, then α is known as a monomorphism. A homomorphism α is known as isomorphism if it is both epimorphism and monomorphism. An isomorphism α is known as an automorphism if $\boldsymbol{R} = \boldsymbol{S}$. If $\alpha : \boldsymbol{R} \to \boldsymbol{S}$ is an isomorphism, then we say that $\boldsymbol{R}$ is isomorphic to $\boldsymbol{S}$ and write

$\boldsymbol{R} \cong \boldsymbol{S}$.

Theorem 2.10

Let $(\boldsymbol{R}, +', \cdot')$ and $(\boldsymbol{S}, +'', \cdot'')$ be two LA-rings. Let $\boldsymbol{I}$ be an ideal of $\boldsymbol{R}$ and $\boldsymbol{J}$ an ideal of $\boldsymbol{S}$. Further let $\overline{I} = \{(r, 0_{\boldsymbol{S}}) : r \in \boldsymbol{I}\}$ and $\overline{J} = \{(0_{\boldsymbol{R}}, s) : s \in \boldsymbol{J}\}$, then

- $\overline{I}$ and $\overline{J}$ are ideals of $\boldsymbol{R} \times \boldsymbol{S}$.
- $\overline{I} \cap \overline{J} = \{(0_{\boldsymbol{R}}, 0_{\boldsymbol{S}})\}$.
- $\overline{I}$ is isomorphic to $\boldsymbol{I}$ and $\overline{J}$ is isomorphic to $\boldsymbol{J}$.

Proof.

- Given that

$\overline{I} = \{(r, 0_S) : r \in \boldsymbol{I}\}$ and $\overline{J} = \{(0_{\boldsymbol{R}}, s) : s \in \boldsymbol{J}\}$.

As $0_{\boldsymbol{R}} \in \boldsymbol{I} \Rightarrow (0_{\boldsymbol{R}}, 0_S) \in \overline{I} \Rightarrow \overline{I}$ is non-empty.

Now Let $c, d \in \overline{I} \Rightarrow c = (r_1, 0_S)$ and $d = (r_2, 0_S)$ for some $r_1, r_2 \in \boldsymbol{I}$. Then

$c - d = (r_1, 0_S) - (r_2, 0_S)$

$= (r_1, 0_S) + (-' r_2, -'' 0_S)$

$= (r_1 +' (-' r_2), 0_S +'' (-'' 0_S))$

$= (r_1 -' r_2, 0_S) \in \overline{I}$ $\qquad\qquad \because r_1 -' r_2 \in \boldsymbol{I}$.

Now let $z \in \boldsymbol{R} \times \boldsymbol{S} \Rightarrow z = (r, s)$ for some $r \in \boldsymbol{R}$ and $s \in \boldsymbol{S}$. Now

$z \cdot c = (r, s) \cdot (r_1, 0_S) = (r \cdot' r_1, s \cdot'' 0_S) = (r \cdot' r_1, 0_S) \in \overline{I}$ $\qquad \because r \cdot' r_1 \in \boldsymbol{I}$.

It follows that $\overline{I}$ is a left ideal. In the same way we may show that $\overline{I}$ is a right ideal.

We now show that $\overline{J}$ is an ideal of $\boldsymbol{R} \times \boldsymbol{S}$.

As $0_S \in \boldsymbol{J} \Rightarrow (0_{\boldsymbol{R}}, 0_S) \in \overline{J} \Longrightarrow \overline{J}$ is non-empty.

Let $m, n \in \overline{J} \Rightarrow m = (0_{\boldsymbol{R}}, s_1)$ and $n = (0_{\boldsymbol{R}}, s_2)$ for some $s_1, s_2 \in \boldsymbol{J}$. Then

$m - n = (0_{\boldsymbol{R}}, s_1) - (0_{\boldsymbol{R}}, s_2) = (0_{\boldsymbol{R}}, s_1) + (-' 0_{\boldsymbol{R}}, -'' s_2)$

$= (0_{\boldsymbol{R}} +' (-' 0_{\boldsymbol{R}}), s_1 +'' (-'' s_2))$

$= (0_{\boldsymbol{R}}, s_1 -'' s_2) \in \overline{J}$ $\qquad\qquad \because s_1 -'' s_2 \in \boldsymbol{J}$.

Now let $z \in \boldsymbol{R} \times \boldsymbol{S} \Rightarrow z = (r, s)$ for some $r \in \boldsymbol{R}$ and $s \in \boldsymbol{S}$. Now

$z \cdot m = (r, s) \cdot (0_{\boldsymbol{R}}, s_1) = (r \cdot' 0_{\boldsymbol{R}}, s \cdot'' s_1) = (0_{\boldsymbol{R}}, s \cdot'' s_1) \in \overline{J}$ $\qquad \because s \cdot'' s_1 \in \boldsymbol{J}$.

It follows that $\overline{J}$ is a left ideal. Similarly we can show that $\overline{J}$ is a right ideal.

- Let $m \in \overline{I} \cap \overline{J} \Rightarrow m \in \overline{I}$ and $m \in \overline{J} \Rightarrow m = (0_{\boldsymbol{R}}, s)$ and $m = (r, 0_S)$ for some $r \in \boldsymbol{I}$ and $s \in \boldsymbol{J} \Rightarrow (0_{\boldsymbol{R}}, s) =$

$(r, 0_S) \Rightarrow r = 0_R$ and $s = 0_S \Rightarrow m = (0_R, 0_S) \Rightarrow \overline{I} \cap \overline{J} = \{(0_R, 0_S)\}$.

- To show that $\overline{I} \cong \boldsymbol{I}$, we define a map $\phi : \boldsymbol{I} \longrightarrow \overline{I}$ by $(r)\phi = (r, 0_S)\ \forall\ r \in \boldsymbol{I}$.

Well-defined:

Let $r_1, r_2 \in \boldsymbol{I}$ be such that $r_1 = r_2 \Rightarrow (r_1, 0_S) = (r_2, 0_S) \Rightarrow (r_1)\phi = (r_2)\phi$.

Onto:

By definition.

One-One:

Let $r_1, r_2 \in \boldsymbol{I}$ be such that $(r_1)\phi = (r_2)\phi \Rightarrow (r_1, 0_S) = (r_2, 0_S) \Rightarrow r_1 = r_2$.

Homomorphism:

Choose $r_1, r_2 \in \boldsymbol{I}$, then

$$
\begin{aligned}
(r_1 +' r_2)\phi &= (r_1 +' r_2, 0_S) \\
&= (r_1 +' r_2, 0_S +'' 0_S) \\
&= (r_1, 0_S) + (r_2, 0_S) \\
&= (r_1)\phi + (r_2)\phi
\end{aligned}
$$

and

$$
\begin{aligned}
(r_1 \cdot' r_2)\phi &= (r_1 \cdot' r_2, 0_S) \\
&= (r_1 \cdot' r_2, 0_S \cdot'' 0_S) \\
&= (r_1, 0_S) \cdot (r_2, 0_S) \\
&= (r_1)\phi \cdot (r_2)\phi.
\end{aligned}
$$

Thus, $\overline{I}$ is isomorphic to $\boldsymbol{I}$.

Next we show that $\overline{J}$ is isomorphic to $\boldsymbol{J}$. For this, we define a map $\alpha : \boldsymbol{J} \longrightarrow \overline{J}$ by $(s)\alpha = (0_R, s)\ \forall\ s \in \boldsymbol{J}$.

Well-defined:

Let $s_1, s_2 \in \boldsymbol{J}$ be such that $s_1 = s_2 \Rightarrow (0_{\boldsymbol{R}}, s_1) = (0_{\boldsymbol{R}}, s_2) \Rightarrow (s_1)\alpha = (s_2)\alpha$.

Onto:

By definition.

One-One:

Let $s_1, s_2 \in \boldsymbol{J}$ be such that $(s_1)\alpha = (s_2)\alpha \Rightarrow (0_{\boldsymbol{R}}, s_1) = (0_{\boldsymbol{R}}, s_2) \Rightarrow s_1 = s_2$.

Homomorphism:

Choose $s_1, s_2 \in \boldsymbol{J}$, then

$$(s_1 +'' s_2)\alpha = (0_{\boldsymbol{R}}, s_1 +'' s_2)$$

$$= (0_{\boldsymbol{R}} +' 0_{\boldsymbol{R}}, s_1 +'' s_2)$$

$$= (0_{\boldsymbol{R}}, s_1) + (0_{\boldsymbol{R}}, s_2)$$

$$= (s_1)\alpha + (s_2)\alpha$$

and

$$(s_1 \cdot'' s_2)\alpha = (0_{\boldsymbol{R}}, s_1 \cdot'' s_2)$$

$$= (0_{\boldsymbol{R}} \cdot' 0_{\boldsymbol{R}}, s_1 \cdot'' s_2)$$

$$= (0_{\boldsymbol{R}}, s_1) \cdot (0_{\boldsymbol{R}}, s_2)$$

$$= (s_1)\alpha \cdot (s_2)\alpha.$$

Thus, $\overline{J}$ is isomorphic to $\boldsymbol{J}$.

Before stating and proving the next result firstly we are going to define kernel of a homomorphism. The following definition has been taken from the source [7].

Definition 2.11

Assume that $\alpha : \boldsymbol{R} \rightarrow \boldsymbol{S}$ is a homomorphism from a left almost ring $\boldsymbol{R}$ into a left almost ring $\boldsymbol{S}$. Then, the kernel of α is denoted by Kerα and is defined as:

$$\text{Ker}\alpha = \{r \in \boldsymbol{R} : (r)\alpha = 0_{\boldsymbol{S}}\}.$$

It can be easily seen that Kerα is an ideal of the LA-ring $\boldsymbol{R}$. Let us state some properties which has been taken from the paper [7].

Theorem 2.12

Let $\boldsymbol{R}$ and $\boldsymbol{S}$ be two LA-rings and $\phi : \boldsymbol{R} \to \boldsymbol{S}$ an epimorphism. Then, $\boldsymbol{R}/\mathrm{Ker}\phi \cong \boldsymbol{S}$.

The above result is known as first isomorphism theorem

Theorem 2.13

Let $(\boldsymbol{R}, +, \cdot)$ be an LA-ring. Let $\boldsymbol{I}$, $\boldsymbol{J}$ be two ideals of $\boldsymbol{R}$. Then, $\boldsymbol{R}/\boldsymbol{I} \cap \boldsymbol{J} \cong \boldsymbol{R}/\boldsymbol{I} \times \boldsymbol{R}/\boldsymbol{J}$.

Proof.

To show that $\boldsymbol{R}/\boldsymbol{I} \cap \boldsymbol{J} \cong \boldsymbol{R}/\boldsymbol{I} \times \boldsymbol{R}/\boldsymbol{J}$, we define a map $\phi : \boldsymbol{R} \longrightarrow \boldsymbol{R}/\boldsymbol{I} \times \boldsymbol{R}/\boldsymbol{J}$ by $(r)\phi = (r + \boldsymbol{I}, r + \boldsymbol{J})$ for all $r \in \boldsymbol{R}$.

First we show that ϕ is well-defined.

Choose $r_1, r_2 \in \boldsymbol{R}$ such that $r_1 = r_2 \Rightarrow r_1 + \boldsymbol{I} = r_2 + \boldsymbol{J} \Rightarrow (r_1)\phi = (r_2)\phi$.

Onto:

By definition.

Homomorphism:

Choose $r_1, r_2 \in \boldsymbol{R}$. Then

$$(r_1 + r_2)\phi = ((r_1 + r_2) + \boldsymbol{I}, (r_1 + r_2) + \boldsymbol{J})$$

$$= ((r_1 + \boldsymbol{I}) + (r_2 + \boldsymbol{I}), (r_1 + \boldsymbol{J}) + (r_2 + \boldsymbol{J}))$$

$$= (r_1 + \boldsymbol{I}, r_1 + \boldsymbol{J}) + (r_2 + \boldsymbol{I}, r_2 + \boldsymbol{J})$$

$$= (r_1)\phi + (r_2)\phi$$

and

$$(r_1 \cdot r_2)\phi = (r_1 \cdot r_2 + \boldsymbol{I}, r_1 \cdot r_2 + \boldsymbol{J})$$

$$= ((r_1 + \boldsymbol{I}) \cdot (r_2 + \boldsymbol{J}), (r_1 + \boldsymbol{I}) \cdot (r_2 + \boldsymbol{J}))$$

$$= (r_1 + \boldsymbol{I}, r_1 + \boldsymbol{I}) \cdot (r_2 + \boldsymbol{J}, r_2 + \boldsymbol{J})$$

$= (r_1)\phi \cdot (r_2)\phi.$

It follows that ϕ is a homomorphism. Thus, by first isomorphism theorem, it follows that

$\boldsymbol{R}/\mathrm{Ker}\phi \cong \boldsymbol{R}/\boldsymbol{I} \times \boldsymbol{R}/\boldsymbol{J}$.

Now we show that $\mathrm{Ker}\phi = \boldsymbol{I} \cap \boldsymbol{J}$.

$\mathrm{Ker}\phi = \{r \in \boldsymbol{R} : (r)\phi = (\boldsymbol{I}, \boldsymbol{J})\}$

$= \{r \in \boldsymbol{R} : (r + \boldsymbol{I}, r + \boldsymbol{J}) = (\boldsymbol{I}, \boldsymbol{J})\}$

$= \{r \in \boldsymbol{R} : r + \boldsymbol{I} = \boldsymbol{I}, r + \boldsymbol{J} = \boldsymbol{J}\}$

$= \{r \in \boldsymbol{R} : r \in \boldsymbol{I}, r \in \boldsymbol{J}\}$

$= \{r \in \boldsymbol{R} : r \in \boldsymbol{I} \cap \boldsymbol{J}\}$

$= \boldsymbol{I} \cap \boldsymbol{J}$.

This completes the proof.

Theorem 2.14

Let $(\boldsymbol{R}, +', \cdot')$ and $(\boldsymbol{S}, +'', \cdot'')$ be two LA-rings. Let $\boldsymbol{I}$ be an ideal of $\boldsymbol{R}$ and $\boldsymbol{J}$ an ideal of $\boldsymbol{S}$, then

- $\boldsymbol{R} \times \boldsymbol{S} \cong \boldsymbol{S} \times \boldsymbol{R}$,
- $\boldsymbol{I} \times \boldsymbol{J} \cong \boldsymbol{J} \times \boldsymbol{I}$.

Proof.

- To show that $\boldsymbol{R} \times \boldsymbol{S} \cong \boldsymbol{S} \times \boldsymbol{R}$, we define a map $\phi : \boldsymbol{R} \times \boldsymbol{S} \longrightarrow \boldsymbol{S} \times \boldsymbol{R}$ by

$$(r, s)\phi = (s, r) \text{ for all } (r, s) \in \boldsymbol{R} \times \boldsymbol{S}.$$

First we show that it is well-defined. Let $(r_1, s_1), (r_2, s_2) \in \boldsymbol{R} \times \boldsymbol{S}$ be such that $(r_1, s_1) = (r_2, s_2) \Rightarrow r_1 = r_2$ and $s_1 = s_2 \Rightarrow (s_1, r_1) = (s_2, r_2) \Rightarrow (r_1, s_1)\phi = (r_2, s_2)\phi$.

One-One:

Let $(r_1, s_1), (r_2, s_2) \in \boldsymbol{R} \times \boldsymbol{S}$ be such that $(r_1, s_1)\phi = (r_2, s_2)\phi \Rightarrow (s_1, r_1) = (s_2, r_2) \Rightarrow s_1 = s_2, r_1 = r_2 \Rightarrow (r_1, s_1) = (r_2, s_2)$.

Onto:

By definition.

Homomorphism:

Let $(r_1, s_1), (r_2, s_2) \in \boldsymbol{R} \times \boldsymbol{S}$, then

$$[(r_1, s_1) + (r_2, s_2)]\phi = (r_1 +' r_2, s_1 +'' s_2)\phi$$

$$= (s_1 +'' s_2, r_1 +' r_2)$$

$$= (s_1, r_1) + (s_2, r_2)$$

$$= (r_1, s_1)\phi + (r_2, s_2)\phi$$

and

$$[(r_1, s_1) \cdot (r_2, s_2)]\phi = (r_1 \cdot' r_2, s_1 \cdot'' s_2)\phi$$

$$= (s_1 \cdot'' s_2, r_1 \cdot' r_2)$$

$$= (s_1, r_1) \cdot (s_2, r_2)$$

$$= (r_1, s_1)\phi \cdot (r_2, s_2)\phi.$$

Thus, it follows that $\boldsymbol{R} \times \boldsymbol{S} \cong \boldsymbol{S} \times \boldsymbol{R}$.

- To show that $\boldsymbol{I} \times \boldsymbol{J} \cong \boldsymbol{J} \times \boldsymbol{I}$, we define $\phi : \boldsymbol{I} \times \boldsymbol{J} \longrightarrow \boldsymbol{J} \times \boldsymbol{I}$ by

$$(i, j)\, \phi = (j, i) \ \forall \ (i, j) \in \boldsymbol{I} \times \boldsymbol{J}.$$

The rest of the proof is similar to part first and is left for the readers.

Theorem 2.15

Let $(\boldsymbol{R}, +', \cdot')$ and $(\boldsymbol{S}, +'', \cdot'')$ be LA-rings with the left identities $0_{\boldsymbol{R}}$ and $0_{\boldsymbol{S}}$ respectively. Then

- $\boldsymbol{R} \times \{0_{\boldsymbol{S}}\} \cong \boldsymbol{R}$,
- $\{0_{\boldsymbol{R}}\} \times \boldsymbol{S} \cong \boldsymbol{S}$.

Proof.

- To show that $\boldsymbol{R} \times \{0_{\boldsymbol{S}}\} \cong \boldsymbol{R}$, we define a map $\phi : \boldsymbol{R} \longrightarrow \boldsymbol{R} \times \{0_{\boldsymbol{S}}\}$ by $(r)\phi = (r, 0_{\boldsymbol{S}})$ for all $r \in \boldsymbol{R}$.

Well-defined:

Let $r_1, r_2 \in \boldsymbol{R}$ be such that $r_1 = r_2 \Rightarrow (r_1, 0_S) = (r_2, 0_S) \Rightarrow (r_1)\phi = (r_2)\phi$.

One-One:

Let $r_1, r_2 \in \boldsymbol{R}$ be such that $(r_1)\phi = (r_2)\phi \Rightarrow (r_1, 0_S) = (r_2, 0_S) \Rightarrow r_1 = r_2$.

Onto:

By definition.

Homomorphism:

Let $r_1, r_2 \in \boldsymbol{R}$, then

$$(r_1 +' r_2)\phi = (r_1 +' r_2, 0_S)$$

$$= (r_1 +' r_2, 0_S +'' 0_S)$$

$$= (r_1, 0_S) + (r_2, 0_S)$$

$$= (r_1)\phi + (r_2)\phi$$

and

$$(r_1 \cdot' r_2)\phi = (r_1 \cdot' r_2, 0_S)$$

$$= (r_1 \cdot' r_2, 0_S \cdot'' 0_S)$$

$$= (r_1, 0_S) \cdot (r_2, 0_S)$$

$$= (r_1)\phi \cdot (r_2)\phi.$$

Thus, it follows that ϕ is a homomorphism.

This completes the proof.

- To show that $\{0_{\boldsymbol{R}}\} \times \boldsymbol{S} \cong \boldsymbol{S}$, we define a map $\alpha : \boldsymbol{S} \longrightarrow \{0_{\boldsymbol{R}}\} \times \boldsymbol{S}$ by $(s)\alpha = (0_{\boldsymbol{R}}, s)$ for all $s \in \boldsymbol{S}$.

The rest of the proof is similar to part first and is left for the readers.

3. Conclusion

In this paper, we have studied left almost rings by using direct products. We have characterized left almost rings by using the properties of direct products but we have seen that problems occur in proving some results. The problem could be removed by exploring some more properties of left almost rings.

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