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On Study K^h - Generalized Birecurrent Finsler Space

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Abstract

In the present paper, a Finsler space F_n whose Cartan's fourth curvature tensor K^i_{jkh} satisfies $K^i_{jkh|\ell|m} = \lambda_\ell K^i_{jkh|m} + b_{\ell m} K^i_{jkh}$, $K^i_{jkh} \neq 0$, where λ_ℓ and $b_{\ell m}$ are non-zero covariant vector field and covariant tensor field of second order, respectively, is introduced and such space is called as K^h–generalized birecurrent Finsler space and denoted briefly by $K^h - GBR - F_n$, we obtained some generalized birecurrent in this space. Also we introduced Ricci generalized birecurrent space.

Keywords: Finsler space; Ricci generalized birecurrent space; generalized birecurrent tensors.

1. Introduction

H. S. Ruse [8] introduced and studied a three dimensional space as space of recurrent curvature. The recurrent of an n–dimensional space was extended to Finsler space by A. Moor [1,2,3] for the first time. Due to different connections of Finsler space, the recurrence of different cuvature tensors have been discussed by R.S. Mishra and H. D. Pande [15] and P. N. Pandey [14]. S. Dikshit [16], discussed a Finsler space in which Cartan's third curvature tensor R_jkh^(i) is birecurrent. M. A. H. Alqufail, F. Y. A. Qasem and M. A. A. Ali [11] discussed a Finsler space in which Cartan's fourth curvature tensor K_jkh^(i) is birecurrent.

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F. Y. A. Qasem [4] discussed a Finsler space in which Cartan's third curvature tensor R_jkh^(i) is generalized and special generalized birecurrent of the first and second kind. F. Y. A. Qasem and A. A. M. Saleem [5,6] discussed a Finsler space for which the h– curvature tensor U_jkh^(i) and Weyl's projective curvature tensor

W_jkh^(i) are generalized birecurrent. N. S. H. Hussein [13] introduced the K^h– recurrent space. Thus, the K^h– recurrent space characterized by (1.1) $K^i_{jkh|\ell} = \lambda_\ell K^i_{jkh}$, $K^i_{jkh} \neq 0$,

where the non–zero covariant vector field λ_ℓ being the recurrence vector field.

M. A. A. Ali [12] discussed the K^h– birecurrent space. Thus, the K^h– birecurrent space is characterized by

(1.2) $$K^i_{jkh|\ell|m} = a_{\ell m} K^i_{jkh}, \qquad K^i_{jkh} \neq 0,$$

where $a_{\ell m}$ is non–zero covariant tensor field of second order is called the birecurrence tensor field.

the metric tensor g_{ij} and its associate metric tensor g^{ij} are covariant constant with respect to the h– covariant derivative

(1.3) $$\text{a) } g_{ij|k} = 0 \quad \text{and} \quad \text{b) } g^{ij}_{|k} = 0 .$$

The h −covariant derivative of the vector y^i, vanish identically, i.e.

(1.4) $$y^i_{|k} = 0.$$

The associate tensor K_{ijkh} of the curvature tensor K^i_{jkh} is given by

(1.5) $$K_{ijkh} := g_{rj} K^r_{ikh}.$$

The Ricci tensor K_{jk} and the curvature scalar K are given by

(1.6) $$\text{a) } K^i_{jki} = K_{jk} \quad \text{and} \quad \text{b) } g^{jk} K_{jk} = K.$$

The curvature tensor K^i_{jkh} satisfies the relation

(1.7) $$K^i_{jkh} y^j = H^i_{kh},$$

The associate tensor R_{ijkh} of the curvature tensor R^i_{jkh} is given by

(1.8) $$R_{ijkh} := g_{rj} R^r_{ikh}$$

The Ricci tensor R_{jk}, the deviation tensor R^r_h and the curvature scalar R are given by

(1.9) a) $R^i_{jki} = R_{jk}$, b) $R^r_{ikh}\, g^{ik} = R^r_k$ and c) $g^{jk}R_{jk} = R$.

Berwald constructed the curvature tensor H^i_{jkh} and the $h(v)$–torsion tensor H^i_{kh} by means of the tensor H^i_k called it by him as deviation tensor [9,10], according to

(1.10) $$H^i_{jkh} = \frac{1}{3}\,\dot{\partial}_j\big(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k\big)$$

and

(1.11) $$H^i_{kh} = \frac{1}{3}\big(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k\big),$$

where

(1.12) $$H^i_h := 2\,\partial_h G^i - \partial_s\, G^i_h\, y^s + 2\, G^i_{hs}\, G^s - G^i_s G^s_h\,.$$

In view of Euler's theorem on homogeneous functions we have the following relation

(1.13) $$H^i_{jk}\, y^j = H^i_k = -H^i_{kj}\, y^j.$$

The contraction of the indices i and h in (1.10), (1.11) and (1.12) yields the following:

(1.14) $$H_{jk} = H^i_{jki},$$

(1.15) $$H_k = H^i_{ki}$$

and

(1.16) $$H = \frac{1}{n-1}\, H^i_i\,,$$

where H_{jk} and H are called h–Ricci tensor [7] and curvature scalar, respectively.

The tensor $H_{jh.k}$ defined by

(1.17) $$H_{jh.k} := g_{ih}\, H^i_{jk}$$

2. An K^h − Generalized Birecurrent Spaces

Let us consider a Finsler space F_n in which Cartan's fourth curvature tensor K^i_{jkh} satisfies

(2.1) $$K^i_{jkh|\ell|m} = \lambda_\ell\, K^i_{jkh|m} + b_{\ell m}\, K^i_{jkh}\ ,\ K^i_{jkh} \neq 0,$$

where λ_ℓ and $b_{\ell m}$ are non-zero covariant vector field and covariant tensor field of second order, respectively.

The space and the tensor satisfying the condition (2.1) will be called K^h–generalized birecurrent space and h–generalized birecurrent tensor, respectively. We shall denote them briefly by K^h– GBR– F_n and h– GBR, respectively.

In view of the conditions (1.1) and (1.2), we may conclude the following results

Theorem 2.1. Every K^h– recurrent space is K^h– birecurrent space, but the converse need not be true.

Differentiation (1.1) covariantly with respect to x^m in the sense of Cartan, we get

$$K^i_{jkh|\ell|m} = \lambda_{\ell|m} K^i_{jkh} + \lambda_\ell K^i_{jkh|m}, \qquad K^i_{jkh} \neq 0$$

which can be written as

$$K^i_{jkh|\ell|m} = \lambda_\ell K^i_{jkh|m} + b_{\ell m} K^i_{jkh}, \qquad K^i_{jkh} \neq 0$$

which it is the condition (1.2), where λ_ℓ and $b_{\ell m} = \lambda_{\ell|m}$ are non–zero covariant vector field and covariant tensor field of second order, respectively.

Theorem 2.2. Every K^h– recurrent space is an K^h– GBR– F_n.

Now, in view of (1.1), the condition (2.1) may written as

$$K^i_{jkh|\ell|m} = \lambda_\ell \lambda_m K^i_{jkh} + b_{\ell m} K^i_{jkh}, \qquad K^i_{jkh} \neq 0$$

which can be written as

$$K^i_{jkh|\ell|m} = a_{\ell m} K^i_{jkh}, \qquad K^i_{jkh} \neq 0,$$

where $a_{\ell m} = \lambda_\ell \lambda_m + b_{\ell m}$ is non–zero covariant tensor field of second order is called the birecurrence tensor field.

Theorem 2.3. In K^h– recurrent space, an K^h– GBR– F_n is K^h– birecurrent space.

Transvecting (2.1) by the metric tensor g_{ip}, using (1.5) and (1.3a), we get

$$\text{(2.2)} \qquad K_{jpkh|\ell|m} = \lambda_\ell K_{jpkh|m} + b_{\ell m} K_{jpkh}.$$

Conversely, the transvection of (2.2) by the associate tensor g^{ip} of the metric tensor g_{ip} yield (2.1). Thus, the condition (2.1) is equivalent to the condition (2.2). Therefore $K^h - GBR - F_n$ may characterized by the condition (2.2).

Thus, we conclude

Theorem 2.4. An $K^h - GBR - F_n$ may characterized by the condition (2.2).

Contracting the indices i and h in (2.1) and using (1.6a), we get

(2.3) $$K_{jk|\ell|m} = \lambda_\ell K_{jk|m} + b_{\ell m} K_{jk}.$$

Showing that the Ricci tensor K_{jk} of $K^h - GBR - F_n$ is generalized birecurrent.

Thus, we conclude

Theorem 2.5. In $K^h - GBR - F_n$, the Ricci tensor is generalized birecurrent.

Transvecting (2.3) by y^k and using (1.4), we get

(2.4) $$K_{j|\ell|m} = \lambda_\ell K_{j|m} + b_{\ell m} K_j,$$

where $K_{jk} y^k = K_j$.

Transvecting (2.1) by g^{jk} and using (1.3b), we get

(2.5) $$K^i_{h|\ell|m} = \lambda_\ell K^i_{h|m} + b_{\ell m} K^i_h ,$$

where $g^{jk} K^i_{jkh} = K^i_h$.

Transvecting (2.3) by g^{jk}, using (1.3b) and (1.6b), we get

(2.6) $$K_{|\ell|m} = \lambda_\ell K_{|m} + b_{\ell m} K.$$

Thus, we conclude

Theorem 2.6. In $K^h - GBR - F_n$, the vector K_j, the deviation tensor K^i_h and the curvature scalar K are all generalized birecurrent.

Transvecting (2.1) by y^j, using (1.4) and (1.7), we get

(2.7) $$H^i_{kh|\ell|m} = \lambda_\ell H^i_{kh|m} + b_{\ell m} H^i_{kh} .$$

Transvecting (2.7) by y^k, using (1.4) and (1.13), we get

(2.8) $$H^i_{h|\ell|m} = \lambda_\ell H^i_{h|m} + b_{\ell m} H^i_h .$$

Contracting the indices i and h in (2.7) and using (1.15), we get

(2.9) $$H_{k|\ell|m} = \lambda_\ell H_{k|m} + b_{\ell m} H_k .$$

Contracting the indices i and h in (2.8) and using (1.16), we get

(2.10) $$H_{|\ell|m} = \lambda_\ell H_{|m} + b_{\ell m} H.$$

Transvecting (2.7) by g_{ji} , using (1.3a) and (1.17), we get

(2.11) $$H_{kj.h|\ell|m} = \lambda_\ell H_{kj.h|m} + b_{\ell m} H_{kj.h} .$$

Thus, we conclude

Theorem 2.7. In $K^h - GBR - F_n$, the h(v)–torsion tensor H^i_{kh} , the deviation tensor H^i_h , the curvature vector H_k, the curvature scalar H and the tensor $H_{kj.h}$ are all generalized bircurrent.

The associate tensor K_{ijkh} of Cartan's fourth curvature tensor K^i_{jkh} and the associate tensor R_{ijkh} of Cartan's third curvature tensor R^i_{jkh} are connected by the identity [7]

(2.12) $$K_{hijk} - K_{ihjk} = 2R_{hijk}.$$

Differentiating (2.12) covariantly with respect to x^ℓ in the sense of Cartan, we get

(2.13) $$K_{hijk|\ell} - K_{ihjk|\ell} = 2R_{hijk|\ell}.$$

Differentiating (2.13) covariantly with respect to x^m in the sense of Cartan and using (2.2), we get

(2.14) $$\lambda_\ell(K_{hijk|m} - K_{ihjk|m}) + b_{lm}(K_{hijk} - K_{ihjk}) = 2R_{hijk|\ell|m}.$$

Putting (2.12) and (2.13) in (2.14), we get

(2.15) $$R_{hijk|\ell|m} = \lambda_\ell R_{hijk|m} + b_{\ell m} R_{hijk}.$$

Transvecting (2.15) by g^{ir}, using (1.3b) and in view of (1.8), we get

(2.16) $$R^r_{hjk|\ell|m} = \lambda_\ell R^r_{hjk|m} + b_{\ell m} R^r_{hjk}.$$

Transvecting (2.16) by g^{hj}, using (1.3b) and (1.9b), we get

(2.17) $$R^r_{k|\ell|m} = \lambda_\ell R^r_{k|m} + b_{\ell m} R^r_k.$$

Contracting the indices r and k in (2.16) and using (1.9a), we get

(2.18) $$R_{hj|\ell|m} = \lambda_\ell R_{hj|m} + b_{\ell m} R_{hj}.$$

Transvecting (2.18) by g^{hj}, using (1.3b) and (1.9c), we get

(2.19) $$R_{|\ell|m} = \lambda_\ell R_{|m} + b_{\ell m} R.$$

Thus, we conclude

Theorem 2.8. In $K^h - GBR - F_n$, Cartan's third curvature tensor R^r_{hjk}, it's associate tensor R_{hijk}, the deviation tensor R^r_k, the Ricci tensor R_{hj} and the scalar R are all generalized bircurrent.

3. Conclusions

(3.1) The space whose defined by condition (2.1) is called K^h–generalized birecurrent Finsler space.

(3.2) Every K^h–recurrent space is an K^h–generalized birecurrent Finsler space.

(3.3) In K^h– recurrent space, an $K^h - GBR - F_n$ is K^h– birecurrent space.

(3.4) In K^h– generalized birecurrent Finsler space , the Ricci tensor is generalized birecurrent.

(3.5) In K^h– generalized birecurrent Finsler space, the vector K_j, the deviation tensor K^i_h and the curvature scalar K are all generalized birecurrent.

(3.6) In K^h– generalized birecurrent Finsler space , the $h(v)$–torsion tensor H^i_{kh}, the deviation tensor H^i_h, the curvature vector H_k, the curvature scalar H and the tensor $H_{kj.h}$ are all generalized bircurrent.

(3.7) In K^h– generalized birecurrent Finsler space, Cartan's third curvature tensor R^r_{hjk}, it's associate tensor R_{hijk}, the deviation tensor R^r_k, the Ricci tensor R_{hj} and the scalar R are all generalized bircurrent.

4. Recommendations

Authors recommend the need for the continuing research and development in Finsler space due to its vital applying importance in other fields.

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