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Classification of Cylindrically Symmetric Marder Type Spacetimes According to Their Proper Homothetic Vector Fields

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Abstract

In this paper cylindrically symmetric Marder type spacetimes are classified according to their proper homothetic vector fields. For the purpose, direct integration and algebraic techniques are used. After thorough investigation, the whole problem is divided into three main cases. Each one of these cases is further reduced into sub cases. It turns out that only in five sub cases proper homothetic vector field exists while the remaining sub cases give contradiction or there exist only Killing vector fields.

Keywords: proper homothetic vector fields; direct integration technique; Lie algebra.

1. Introduction

Symmetries in general relativity are vector fields which preserve some geometrical object on spacetimes. There are many kinds of symmetries that give information about spacetimes.

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Symmetries are useful to find the solution of Einstein Field equation. Einstein's Field equations are given as

$R_{cd} - \frac{1}{2} R g_{cd} + \Lambda g_{cd} = \frac{8\pi G}{c^2} T_{cd}$ where R_{cd} is Ricci tensor, R is Ricci scalar, g_{cd} is the metric of spacetime, Λ is the cosmological constant, c is the speed of light, T_{cd} is energy momentum tensor, and G is the gravitational constant. Also $R_{cd} - \frac{1}{2} R g_{cd} = G_{cd}$ is called the Einstein tensor [1, 5, 8].

Our purpose is to investigate proper homothetic symmetry in Marder type space-times by using direct integration techniques [4]. Here throughout M is a four dimensional manifold and $\nabla_a \nabla_c t^b - \nabla_c \nabla_a t^b = -R^b_{acs} t^s$, where R^b_{acs} is the Riemann curvature tensor. The Lie derivative is denoted by L [5, 19]. Round and square brackets represent the symmetrization and skew- symmetrization respectively [2]. For any vector field Y on a manifold M we have

$$X_{\alpha;\beta} = \frac{1}{2} K_{\alpha\beta} + M_{\alpha\beta} \qquad (1.1)$$

where $K_{\alpha\beta} = K_{\beta\alpha}$ is symmetric tensor and $M_{\alpha\beta} = -M_{\beta\alpha}$ is skew-symmetric tensor [3, 6]. The covariant derivative is denoted by a semicolon. A vector field Y on M is a solution of homothety equation $\underset{Y}{L} g_{ab} = K_{ab} = 2\phi g_{ab}, \phi \in R$ [5, 7] and we can write

$$g_{ab,c} Y^c + g_{bc} Y^c_{,a} + g_{ca} Y^c_{,b} = 2\phi g_{ab}, (a,b,c = 0,1,2,3) \qquad (1.2)$$

The partial derivative is denoted by a comma. If $\phi \neq 0$ then vector field Y is proper homothetic vector field otherwise Killing vector field [5, 9, 10, 11].

2. Main Results

The Marder type space- time with line element is [9-19] $ds^2 = -A^2(t)dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2$, where $A(t), B(t)$ and $C(t)$ are no where zero functions. Using equation (1.2) we get following ten partial differential equations.

$$\frac{A^{\cdot}}{A} X^0 + X^0_{,0} = \phi \qquad (2.1)$$

$$X^1_{,0} - X^0_{,1} = 0 \qquad (2.2)$$

$$B^2 X^2_{,0} - A^2 X^0_{,2} = 0 \qquad (2.3)$$

$$C^2 X^3_{,0} - A^2 X^0_{,3} = 0 \qquad (2.4)$$

$$\frac{A^{\cdot}}{A} X^0 + X^1_{,1} = \phi \qquad (2.5)$$

$$B^2 X^2_{,1} + A^2 X^1_{,2} = 0 \qquad (2.6)$$

$$C^2 X^3_{,1} + A^2 X^1_{,3} = 0 \qquad (2.7)$$

$$\frac{B^{\cdot}}{B} X^0 + X^2_{,2} = \phi \qquad (2.8)$$

$$C^2 X^3_{,2} + B^2 X^2_{,3} = 0 \qquad (2.9)$$

$$\frac{C^{\cdot}}{C} X^0 + X^3_{,3} = \phi \qquad (2.10)$$

Using equations (2.1) to (2.10) with some calculation we get following system

$$\left.\begin{aligned} X^0 &= \int [L^3_t(t,x,y) + L^4_t(t,x,z)]dx + L^5(t,y,z) \\ X^1 &= L^3(t,x,y) + L^4(t,x,z) \\ X^2 &= L^1(t,x,y) + L^2(t,y,z) \\ X^3 &= -\frac{A^2}{C^2}\int L^4_z(t,x,z)dx + L^6(t,y,z) \end{aligned}\right\} \qquad (2.11)$$

Now substituting the above system in equation (2.5) and differentiating w. r. t. y and z, we get $A^{\cdot} L^5_{yz}(t,y,z) = 0.$ There are three possibilities

Case (1). $A^{\cdot} \neq 0$ and $L^5_{yz}(t,y,z) = 0$ Case (2). $A^{\cdot} = 0$ and $L^5_{yz}(t,y,z) \neq 0$

Case (3). $A^{\cdot} = 0$ and $L^5_{yz}(t,y,z) = 0$

We have solved all the three possibilities in turn and it comes out that in Cases (1) and (2) Marder type spacetime does not admit proper homothetic vector fields. Only in Case (3) we get some results for different choices of the metric functions. These results are listed below:

Case 3 (a): After calculation with (2.9) we get solution of the above ten partial differential equations as

$$\left.\begin{array}{lll} X^0 = \phi t + c_{468}, & X^1 = \phi x + c_{462}, & X^2 = (\phi - c_{485})y - c_{441}z + c_{482} \\ X^3 = c_{441}y + (\phi - c_{485})z + c_{479} & & \end{array}\right\} \quad (2.13)$$

Where as $A = c_{295}$, $B = C = c_{486}(\phi t + c_{468})^{\frac{c_{485}}{\phi}}$. Therefore Marder type space time becomes

$$ds^2 = -c_{295}^2 dt^2 + c_{295}^2 dx^2 + c_{486}^2 (\phi t + c_{468})^{2\frac{c_{485}}{\phi}} (dy^2 + dz^2) \quad (2.14)$$

Case 3(b): In this case we get the solution of homothetic equations as follows:

$$\left. X^0 = \phi t + c_{468}, \quad X^1 = \phi x + c_{462}, \quad X^2 = (\phi - c_{485})y + c_{482}, \quad X^3 = (\phi - c_{483})z + c_{479} \right\} \quad (2.16)$$

Where as $A = c_{295}$, $B = c_{486}(\phi t + c_{468})^{\frac{c_{485}}{\phi}}$, $C = c_{484}(\phi t + c_{468})^{\frac{c_{483}}{\phi}}$. Therefore given space- time becomes

$$ds^2 = -c_{295}^2 dt^2 + c_{295}^2 dx^2 + c_{486}^2 (\phi t + c_{468})^{2\frac{c_{485}}{\phi}} dy^2 + c_{484}^2 (\phi t + c_{468})^{2\frac{c_{483}}{\phi}} dz^2 \quad (2.17)$$

Case 3(c): In this case solution of equations (2.1) to (2.10) is

$$\left.\begin{array}{ll} X^0 = \phi t + c_{511}, & X^1 = \phi x + c_{450}z + c_{513}, \\ X^2 = -c_{514}y + \phi y + c_{519}, & X^3 = -c_{450}\dfrac{c_{295}^2}{c_{501}^2}x + \phi z + c_{509} \end{array}\right\} \quad (2.22)$$

Where as $A = c_{295}$, $B = c_{518}\,(\phi t + c_{511})^{\frac{c_{514}}{\phi}}$, $C = c_{501}$. Therefore given space time becomes

$$ds^2 = -c_{295}^2 dt^2 + c_{295}^2 dx^2 + c_{518}^2 (\phi t + c_{511})^{2\frac{c_{514}}{\phi}} dy^2 + c_{501}^2 dz^2 \quad (2.23)$$

Case 3(d): in this case after lengthy calculation we get solution as

$$\left. X^0 = \phi t + c_{573}, \quad X^1 = \phi x - c_{556}\frac{c_{520}^2}{c_{295}^2}y + c_{569}, \quad X^2 = c_{556}x + \phi y + c_{564}, \quad X^3 = (\phi - c_{576})z + c_{574} \right\}$$

(2.25)

Where as $A = c_{295}, B = c_{520}, C = c_{577}(\phi t + c_{573})^{\frac{c_{576}}{\phi}}$. Given spacetime becomes

$$ds^2 = -c_{295}^2 dt^2 + c_{295}^2 dx^2 + c_{520}^2 dy^2 + c_{577}^2 (\phi t + c_{573})^{2\frac{c_{576}}{\phi}} dz^2$$

(2.26)

3. Conclusions

In this paper we classified Marder type spacetimes according to its homothetic vector fields. After examining all possibilities we found four cases where proper homothetic vector field exists which is shown in systems (2.13), (2.16), (2.22) and (2.25) where as their respective spacetime metrics are given in (2.14), (2.17), (2.23) and (2.26). It is shown that Marder type spacetimes admit eight, seven or six linearly independent homothetic vector fields and in which only one is proper homothetic vector field. Therefore the generators and their Lie algebras for each case are given as under:

For solution (2.13) $X_1 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = \frac{\partial}{\partial x}$, $X_4 = -y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}$, $X_5 = -z\frac{\partial}{\partial z}, X_6 = \frac{\partial}{\partial y}, X_7 = y\frac{\partial}{\partial z}, X_8 = \frac{\partial}{\partial z}$. The Lie algebra is $[X_1, X_2] = -X_2, [X_1, X_3] = -X_3$, $[X_1, X_6] = -X_6$, $[X_1, X_8] = -X_8$, $[X_4, X_6] = X_6 = [X_5, X_8]$, $[X_4, X_8] = X_8 = [X_6, X_7]$, $[X_5, X_7] = -zX_8 + yX_6$ $[X_\alpha, X_\beta] = 0, otherwise$

For solution (2.16) $X_1 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = \frac{\partial}{\partial x}$, $X_4 = -y\frac{\partial}{\partial y}$, $X_5 = \frac{\partial}{\partial y}$, $X_6 = -z\frac{\partial}{\partial z}$, $X_7 = \frac{\partial}{\partial z}$ The Lie algebra is $[X_1, X_2] = -X_2$, $[X_1, X_3] = -X_3$, $[X_1, X_5] = -X_5$, $[X_1, X_7] = -X_7, [X_4, X_5] = X_5, [X_6, X_7] = -X_6$ $[X_\alpha, X_\beta] = 0, otherwise$.

For solution (2.22) generators are $X_1 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = z\frac{\partial}{\partial x}$, $X_4 = \frac{\partial}{\partial x}$, $X_5 = -y\frac{\partial}{\partial y}$, $X_6 = \frac{\partial}{\partial y}$, $X_7 = -x\frac{\partial}{\partial z}$, $X_8 = \frac{\partial}{\partial z}$. The Lie algebra is $[X_1, X_2] = -X_2$, $[X_1, X_4] = -X_4 = [X_3, X_8]$, $[X_1, X_6] = -X_6$, $[X_1, X_8] = -X_8 = [X_4, X_7]$, $[X_3, X_7] = -zX_8 + xX_4, [X_5, X_6] = X_6$ $[X_\alpha, X_\beta] = 0, otherwise$.

For solution (2.25) $X_1 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = -y\frac{\partial}{\partial x}$, $X_4 = \frac{\partial}{\partial x}$, $X_5 = x\frac{\partial}{\partial y}$, $X_6 = \frac{\partial}{\partial y}$, $X_7 = -z\frac{\partial}{\partial z}$, $X_8 = \frac{\partial}{\partial z}$. The Lie algebra is $[X_1, X_2] = -X_2$, $[X_1, X_6] = -X_6$, $[X_1, X_8] = -X_8$, $[X_3, X_5] = -yX_6 + xX_4$, $[X_3, X_6] = X_4$, $[X_4, X_5] = X_6$, $[X_7, X_8] = X_8$, $[X_\alpha, X_\beta] = 0, \; otherwise$.

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