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On a Generalized H^h- Birecurrent Finsler Space

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Abstract

In the present paper, a Finsler space whose curvature tensor H^i_{jkh} satisfies $H^i_{jkh|\ell|m} = a_{\ell m}H^i_{jkh} + b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $H^i_{jkh} \neq 0$, where $a_{\ell m}$ and $b_{\ell m}$ are non-zero covariant tensor fields of second order called recurrence tensor fields, is introduced, such space is called as a generalized H^h −birecurrent Finsler space . The associate tensor H_{jrkh} of Berwald curvature tensor H^i_{jkh} , the torsion tensor H^i_{kh} ,the deviation tensor H^i_h, the Ricci tensor H_{jk}, the vector H_k and the scalar curvature H of such space are non-vanishing. Under certain conditions, a generalized H^h −birecurrent Finsler space becomes Landsberg space . Some conditions have been pointed out which reduce a generalized H^h −birecurrent Finsler space $F_n(n > 2)$ into Finsler space of scalar curvature.

Keywords: Finsler space; Generalized H^h −birecurrent Finsler space; Ricci tensor; Landsberg space; Finsler space of scalar curvature.

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1. Introduction

H. S. Ruse [6] considered a three dimensional Riemannian space having the recurrent of curvature tensor and he called such space as Riemannian space of recurrent curvature. This idea was extended to n-dimensional Riemannian and non- Riemannian space by A.G. Walker [2], Y. C. Wong [18] ,Y.C. Wong and K. Yano [19] and others .

This idea was extended to Finsler spaces by A. Moor [3] for the first time . Due to different connections of Finsler space, the recurrent of different curvature tensor have been discussed by, R.B. Misra [12] , R.B. Misra and F. M. Meher [13] , B.B.Sinha and S.P.Singh [4] , P. N. pandey and R.B.Misra [10], R.S.D.Dubey and A.K. Srivastara [14] , R.Verma [16] , S. Dikshit [17] , and others. S. Dikshit discussed different Finsler space with birecurrent of Cartan`s curvature tensor, birecurrent of its associate tensor and indicatrix with respect to Berwald`s and Cartan`s connections . F.Y.A.Qasem and A.A.M.Saleem [5] discussed more general Finsler space for the hv −curvature tensor U^i_{jkh} satisfies the birecurrence property with respect to Berwald's coefficient G^i_{jk} and they called it UBR- Finsler space. A.A.M.Saleem [1] discussed C^h– generalized birecurrent Finsler space and U –special generalized birecurrent Finsler space. P.N.pandey, S.Saxena and A.Goswami [11] interduced a generalized H-recurrent Finsler space.

Let F_n be An n-dimensional Finsler space equipped with the metric function a $F(x, y)$ satisfying the request conditions [6] .

The vectors y_i , $y^i$and the metric tensor g_{ij} satisfies the following relations

(1.1) $a)\ \ y_i\, y^i = F^2 \qquad b)\ \ g_{ij} = \dot{\partial}_i y_j = \dot{\partial}_j y_i \qquad c)\ \ y_{i|k} = 0$

$d)\ \ y^i_{\ |k} = 0$ and $e)\ \ g_{ij|k} = 0$.

Thus the unit vector l^i and the associate vector l_i is defined by

(1.2) $a)\ \ l^i = \frac{y^i}{F} \qquad b)\ \ l_i = g_{ij} l^j = \dot{\partial}_i F = \frac{y_i}{F}$.

The two processes of covariant differentiation, defined above commute with the partial

(1.3) $a)\ \ \dot{\partial}_j (X^i_{\ |k}) - (\dot{\partial}_j X^i)_{|k} = X^r (\dot{\partial}_j \Gamma^{*i}_{rk}) - (\dot{\partial}_r X^i) P^r_{jk}$,

$b)\ \ P^r_{jk} = (\dot{\partial}_j \Gamma^{*r}_{hk}) y^h = \Gamma^{*r}_{jhk}\, y^h$,

$c)\ \ \Gamma^{*i}_{jkh}\, y^h = G^i_{jkh}\, y^h = 0$,

$d)\ \ P^i_{jk} y^j = 0$,

$e)\ \ g_{ir} P^i_{kh} = P_{rkh}$.

The tensor H^i_{jkh} satisfies the relation

(1.4) $H^{\,i}_{jkh}\, y^{\,j} = H^{\,i}_{kh}$.

(1.5) $H^{\,i}_{jkh} = \dot{\partial}_j H^{\,i}_{kh}$.

The deviation tensor $H^{\,i}_{k}$ is positively homogeneous of degree two in $y^{\,i}$ and satisfies

(1.6) $H^{\,i}_{hk}\, y^{\,h} = H^{\,i}_{k}$,

(1.7) $H^{i}_{k}\, y^{k} = 0$,

(1.8) $H_{\,jk} = H^{\,i}_{jki}$,

(1.9) $H_{\,k} = H^{\,i}_{ki}$,

and

(1.10) $H = \frac{1}{n-1}\, H^{\,i}_{i}$.

where $H_{\,jk}$ and H are called *h-Ricci tensor* [9] and *curvature scalar* respectively. Since contraction of the indices does not affect the homogeneity in y^{i}, hence the tensors H_{rk} , H_{r} and the scalar H are also homogeneous of degree zero, one and two in y^{i} respectively . The above tensors are also connected by

(1.11) $H_{\,jk}\, y^{j} = H_{\,k}$,

(1.12) $H_{\,jk} = \dot{\partial}_j\, H_{\,k}$,

(1.13) $H_{\,k}\, y^{\,k} = (n-1)H$.

The tensors $H^{\,i}_{h}$, $H^{\,i}_{kh}$ and $H^{\,i}_{jkh}$ also satisfy the following :

(1.14) $H^{\,i}_{kh} = \dot{\partial}_k\, H^{\,i}_{h}$,

(1.15) $g_{\,ij}\, H^{\,i}_{k} = g_{\,ik}\, H^{\,i}_{j}$.

The associate tensor $H_{\,ijkh}$ of Berwald curvature tensor $H^{\,i}_{jkh}$ is given by

(1.16) $H_{\,ijkh} = g_{\,rj}\, H^{\,r}_{ikh}$.

The necessary and sufficient condition for a Finsler space $F_n (n > 2)$ to be a Finsler space of scalar curvature is given by

(1.17) $H^{i}_{h} = F^2 R(\,\delta^{i}_{h} - l^{\,i}\, l_{h}\,)$.

A Finsler space F_n is said to be Landsberg space if satisfies

(1.18) $y_r G^{r}_{jkh} = -2C_{jkh|m} y^{m} = -2P_{jkh} = 0.$

2. Generalized H^h −Birecurrent Finsler Space

Let us consider a Finsler space F_n whose Berwald curvature tensor H^i_{jkh} satisfies

(2.1) $H^i_{jkh|\ell} = \lambda_\ell H^i_{jkh} + \mu_\ell(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk})$, $H^i_{jkh} \neq 0$, where λ_ℓ and μ_ℓ are non-zero covariant vector fields and called the recurrence vector fields. Such space called a generalized H^h- recurrent Finsler space.

Differentiating (*2.1*) covariantly with respect to x^m in the sense of Cartan and using *(1.1.e*), we get

(2.2) $$H^i_{jkh|\ell|m} = \lambda_{\ell|m} H^i_{jkh} + \lambda_\ell H^i_{jkh|m} + \mu_{\ell|m}(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}).$$

Using (*2.1*) in (*2.2*), we get

$$H^i_{jkh|\ell|m} = (\lambda_{\ell|m} + \lambda_\ell\, \lambda_m) H^i_{jkh} + (\lambda_\ell \mu_m + \mu_{\ell|m})(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}),$$

which can be written as

(2.3) $$H^i_{jkh|\ell|m} = a_{\ell m} H^i_{jkh} + b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk}),\ H^i_{jkh} \neq 0\ ,$$

Where $a_{\ell m} = \lambda_{\ell|m} + \lambda_\ell\, \lambda_m$ and $b_{\ell m} = \lambda_\ell \mu_m + \mu_{\ell|m}$ are non-zero covariant tensor fields of second order and called recurrence tensor fields.

Definition 2.1. If Berwald curvature tensor H^i_{jkh} of a Finsler space satisfying the condition(*2.3*), where $a_{\ell m}$ and $b_{\ell m}$ are non-zero covariant tensor fields of second order , the space and the tensor will be called generalized $H^h -$ birecurrent Finsler space and generalized $h -$ birecurrent tensor, respectively, we shall denote such space and tensor briefly by $GH^h - BR - F_n$ and $Gh\text{-}BR$, respectively .

However, if we start from condition(*2.3)*, we cannot obtain the condition(*2.1*), we may conclude

Theorem 2.1. Every generalized $H^h -$ recurrent Finsler space is generalized $H^h -$ birecurrent Finsler space**,** but the converse need not be true.

Transvecting (*2.3*) by the metric tensor g_{ir} , using (*1.1e*) and (*1.16*) ,we get

(2.4) $$H_{jrkh|\ell|m} = a_{\ell m} H_{jrkh} + b_{\ell m}(g_{kr} g_{jh} - g_{hr} g_{jk}).$$

Transvecting (*2.3*) by y^j , using (*1.1d*) and(*1.4*), we get

(2.5) $$H^i_{kh|\ell|m} = a_{\ell m} H^i_{kh} + b_{\ell m}(\delta^i_k y_h - \delta^i_h y_k).$$

Further transvecting (*2.5*) by y^k , using *(1.1d*) and *(1.6*), we get

(2.6) $$H^i_{h|\ell|m} = a_{\ell m} H^i_h + b_{\ell m}(y^i y_h - \delta^i_h F^2).$$

Thus we have

Theorem 2.2. In $GH^h - BR - F_n$,the associate tensor H_{jrkh} of Berwald curvature tensor H^i_{jkh} , the torsion tensor H^i_{kh} and the deviation tensor H^i_h are non- vanishing.

Contracting the indices i and h in equations(*2.3*), *(2.5*) and (*2.6*) and using(*1.8*), (*1.9)*, (*1.10*) and (*1.1 a*), we get

(2.7) $$H_{jk|\ell|m} = a_{\ell m} H_{jk} + (1-n) b_{\ell m} g_{jk} \,.$$

(2.8) $$H_{k|\ell|m} = a_{\ell m} H_k + (1-n) b_{\ell m} y_k \,.$$

(2.9) $$H_{|\ell|m} = a_{\ell m} H - b_{\ell m} F^2 \,.$$

Thus, we conclude

Theorem 2.3. In $GH^h - BR - F_n$ the Ricci tensor H_{jk}, the curvature vector H_k and the scalar curvature H are non- vanishing.

Differentiating (*2.8*) partially with respect to y^j, using (*1. 12*) and *(1.1b*), we get

(2.10) $$\dot\partial_j\left(H_{k|\ell|m}\right) = \left(\dot\partial_j a_{\ell m}\right) H_k + a_{\ell m} H_{jk} + (1-n)\left(\dot\partial_j b_{\ell m}\right) y_k$$
$$+(1-n) b_{\ell m}\, g_{jk} \,.$$

Using the commutation formula exhibited by(*1. 3a*) for ($H_{k|\ell}$) and using(*1.12*), we get

(2.11) $$(\dot\partial_j H_{k|\ell})_{|m} - H_{r|\ell}\left(\dot\partial_j \Gamma^{*r}_{km}\right) - H_{k|r}\left(\dot\partial_j \Gamma^{*r}_{\ell m}\right) - \left(\dot\partial_r H_{k|\ell}\right) P^r_{jm}$$
$$= \left(\dot\partial_j a_{\ell m}\right) H_k + a_{\ell m} H_{jk} + (1-n)\left(\dot\partial_j b_{\ell m}\right) y_k + (1-n) b_{\ell m} g_{jk} \,.$$

Again using commutation formula exhibited by *(1.3a*) for (H_k) in(*2.11*) , we get

(2.12) $$\left\{(\dot\partial_j H_k)_{|\ell} - H_r\left(\dot\partial_j \Gamma^{*r}_{\ell k}\right) - (\dot\partial_r H_k) P^r_{j\ell}\right\}_{|m} - H_{r|\ell}\left(\dot\partial_j \Gamma^{*r}_{km}\right)$$
$$-H_{k|r}\left(\dot\partial_j \Gamma^{*r}_{\ell m}\right) - \left\{(\dot\partial_r H_k)_{|\ell} - H_s\left(\dot\partial_r \Gamma^{*s}_{\ell k}\right) - (\dot\partial_s H_k) P^s_{r\ell}\right\} P^r_{jm}$$
$$= \left(\dot\partial_j a_{\ell m}\right) H_k + a_{\ell m} H_{jk} + (1-n)\left(\dot\partial_j b_{\ell m}\right) y_k + (1-n) b_{\ell m} g_{jk} \,.$$

Using *(1.12*) and *(2.7*) in *(2.12*), we get

(2.13) $$\left\{-H_r\left(\dot\partial_j \Gamma^{*r}_{\ell k}\right) - (H_{kr}) P^r_{j\ell}\right\}_{|m} - H_{r|\ell}\left(\dot\partial_j \Gamma^{*r}_{km}\right)$$
$$-H_{k|r}\left(\dot\partial_j \Gamma^{*r}_{\ell m}\right) - \left\{H_{kr\,|\ell} - H_s\left(\dot\partial_r \Gamma^{*s}_{\ell k}\right) - H_{ks} P^s_{r\ell}\right\} P^r_{jm}$$
$$= \left(\dot\partial_j a_{\ell m}\right) H_k + (1-n)\left(\dot\partial_j b_{\ell m}\right) y_k \,.$$

Transvecting (*2.13*) by y^k , using (*1.1d*) , *(1.13*),(*1.3b*) and (*1.1a*), we ge

$$-2H_{r|\ell}P^r_{jm} - (n-1)H_{|r}\left(\dot{\partial}_j\Gamma^{*r}_{\ell m}\right) = (n-1)\left(\dot{\partial}_j a_{\ell m}\right)H - (n-1)\left(\dot{\partial}_j b_{\ell m}\right)F^2.$$

Which can be written as

(2.14) $$\left(\dot{\partial}_j b_{\ell m}\right) = \frac{(\dot{\partial}_j a_{\ell m})H}{F^2}\ .$$

if and only if

(2.15) $$-2H_{r|\ell}P^r_{jm} - (n-1)H_{|r}\left(\dot{\partial}_j\Gamma^{*r}_{\ell m}\right) = 0\ .$$

If the tensor $a_{\ell m}$ is independent of y^i, the equation (*2.14*) shows that the tensor $b_{\ell m}$ is also independent of y^i. Conversely , if the tensor $b_{\ell m}$ is independent of y^i, we get $H\dot{\partial}_j a_{\ell m} = 0$. In view of theorem2.3, the condition $H\dot{\partial}_j a_{\ell m} = 0$ implies $\dot{\partial}_j a_{\ell m} = 0$, ,i.e. the covariant tensor $a_{\ell m}$ is also independent of y^i. This leads to

Theorem 2.4. The covariant tensor $b_{\ell m}$ is independent of the directional arguments if the covariant tensor $a_{\ell m}$ is independent of directional arguments if and only if equation (*2.15)* holds.

Suppose the tensor $a_{\ell m}$ is not independent of y^i**,** then *(2.13*) and(*2.14***)** together imply

(2.16) $$\left\{-H_r\left(\dot{\partial}_j\Gamma^{*r}_{\ell k}\right) - (H_{kr})P^r_{j\ell}\right\}_{|m} - H_{r|\ell}\left(\dot{\partial}_j\Gamma^{*r}_{km}\right)$$
$$-H_{k|r}\left(\dot{\partial}_j\Gamma^{*r}_{\ell m}\right) - \left\{H_{kr|\ell} - H_s(\dot{\partial}_r\Gamma^{*s}_{\ell k}) - H_{ks}P^s_{r\ell}\right\}P^r_{jm}$$
$$= \left(\dot{\partial}_j a_{\ell m}\right)\left[H_k - \frac{(n-1)}{F^2}Hy_k\right].$$

Transvecting *(2.16)* by y^m and using (*1.1d)* , (*1.3c)* and *(1.3d*), we get

(*2.17*) $$\left\{-H_r\left(\dot{\partial}_j\Gamma^{*r}_{\ell k}\right) - (H_{kr})P^r_{j\ell}\right\}_{|m} y^m = \left(\dot{\partial}_j a_\ell - a_{j\ell}\right)\left(H_k - \frac{(n-1)}{F^2}Hy_k\right).$$

where $a_{\ell m}y^m = a_\ell$

if

(2.18) $\left\{-H_r\left(\dot{\partial}_j\Gamma^{*r}_{\ell k}\right) - (H_{kr})P^r_{j\ell}\right\}_{|m} y^m = 0$, equation *(2.17)* implies at least one of the following conditions

(2.19) a) $a_{j\ell} = \dot{\partial}_j a_\ell$, b) $H_k = \frac{(n-1)}{F^2}Hy_k$

Thus, we have

Theorem 2.5. In $GH^h - BR - F_n$ for which the covariant tensor $a_{\ell m}$ is not independent of the directional arguments and if condition (*2.18)* holds, at least one of the conditions (*2.19a)* and (*2.19b*) hold.

Suppose (*2.19b*) holds equation (*2.16)* implies

(2.20) $\left\{-\frac{(n-1)}{F^2}Hy_r\dot{\partial}_j\Gamma^{*r}_{\ell k} - H_{kr}P^r_{j\ell}\right\}_{|m} - \left\{\frac{(n-1)}{F^2}Hy_r\right\}_{|\ell}\dot{\partial}_j\Gamma^{*r}_{km}$

$-\left\{\frac{(n-1)}{F^2}Hy_k\right\}_{|r}\dot{\partial}_j\Gamma^{*r}_{\ell m} - H_{kr|\ell}P^r_{jm} - \frac{(n-1)}{F^2}Hy_s\left(\dot{\partial}_r\Gamma^{*s}_{\ell k}\right)P^r_{jm}$

$- H_{ks}P^s_{r\ell}P^r_{jm} = 0$.

Transvecting *(2.20)* by y^j , using (*1.1d)*, (*1.3b*) and (*1.3d*), we get

(2.21) $\left\{\frac{(n-1)}{F^2}Hy_r\mathrm{P}^r_{\ell k}\right\}_{|m} + \left\{\frac{(n-1)}{F^2}Hy_r\right\}_{|\ell}P^r_{km} + \left\{\frac{(n-1)}{F^2}Hy_k\right\}_{|r}\mathrm{P}^r_{\ell m} = 0$.

Thus, we have

Theorem 2.6. In $GH^h - BR - F_n$, we have the identity (2.21) provided (2.19b).

Transvecting (2.21) by the metric tensor g_{rj} , using (1.1e) and (1.3e), we get

(2.22) $\left\{\frac{(n-1)}{F^2}Hy_rP_{j\ell k}\right\}_{|m} + \left\{\frac{(n-1)}{F^2}Hy_r\right\}_{|\ell}P_{jkm} + \left\{\frac{(n-1)}{F^2}Hy_k\right\}_{|r}P_{j\ell m} = 0.$

By using *(1.1.c)* , equation *(1.22)* can be written as

$$y_r(HP_{j\ell k})_{|m} + y_rH_{|\ell}P_{jkm} + y_kH_{|r}P_{j\ell m} = 0\ .$$

In view of theorem2.3 , we have

(2. 23) $P_{j\ell m} = 0$.

if and only if

(2.24) $y_r(HP_{j\ell k})_{|m} + y_rH_{|\ell}P_{jkm} = 0$.

Therefore the space is Landsberg space.

Thus, we have

Theorem 2.7. An $GH^h - BR - F_n$ is Landsberg space if and only if conditions *(2.24)* and (*2.19b)* hold good.

If the covariant tensor $a_{j\ell} \neq \dot{\partial}_j a_\ell$, in view of theorem2.5, (*2.19b)* holds good. In view of this fact, we may rewrite theorem 2.7 in the following form

Theorem 2.8. An $GH^h - BR - F_n$ is necessarily Landsberg space if and only conditions (*2.24*) and (*2.19b*) hold good and provided $a_{j\ell} \neq \dot{\partial}_j a_\ell$.

Differentiating *(2.5)* partially with respect to y^j , using *(1.5*) and (*1.1b*), we get

$$(2.25)\qquad \dot{\partial}_j\left(H^i_{kh|\ell|m}\right) = (\dot{\partial}_j a_{\ell m})H^i_{kh} + a_{\ell m}H^i_{jkh} + (\dot{\partial}_j b_{\ell m})(\delta^i_k y_h - \delta^i_h y_k)$$

$$+b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

Using commutation formula exhibited by (*1.3b*) for ($H^i_{kh|\ell}$) in (*2.25*), we get

$$(2.26)\qquad \left\{\dot{\partial}_j\left(H^i_{kh|\ell}\right)\right\}_{|m} + H^r_{kh|\ell}(\dot{\partial}_j\Gamma^{*i}_{rm}) - H^i_{rh|\ell}(\dot{\partial}_j\Gamma^{*r}_{km}) - H^i_{rk|\ell}(\dot{\partial}_j\Gamma^{*r}_{hm})$$

$$-\ H^r_{kh|r}(\dot{\partial}_j\Gamma^{*i}_{m\ell}) - \dot{\partial}_r\left(H^i_{kh|\ell}\right)P^r_{jm} = (\dot{\partial}_j a_{\ell m})H^i_{kh}$$

$$+a_{\ell m}H^i_{jkh} + (\dot{\partial}_j b_{\ell m})(\delta^i_k y_h - \delta^i_h y_k) + b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

Again applying the commutation formula exhibited by (*1.3a*) for (H^i_{kh}) in (*2.26*) and using (*1.5*), we get

$$(2.27)\qquad \left\{H^i_{jkh|\ell} + H^r_{kh}(\dot{\partial}_j\Gamma^{*i}_{r\ell}) - H^i_{rh}(\dot{\partial}_j\Gamma^{*r}_{k\ell}) - H^i_{rk}(\dot{\partial}_j\Gamma^{*r}_{h\ell}) - H^i_{rkh}P^r_{j\ell}\right\}_{|m}$$

$$+H^r_{kh|\ell}(\dot{\partial}_j\Gamma^{*i}_{rm}) - H^i_{rh|\ell}(\dot{\partial}_j\Gamma^{*r}_{km}) - H^i_{rk|\ell}(\dot{\partial}_j\Gamma^{*r}_{hm}) - \ H^i_{kh|r}(\dot{\partial}_j\Gamma^{*r}_{m\ell}) -$$

$$\left\{H^i_{rkh|\ell} + H^s_{kh}(\dot{\partial}_r\Gamma^{*i}_{s\ell}) - H^i_{sh}(\dot{\partial}_r\Gamma^{*s}_{k\ell}) - H^i_{sk}(\dot{\partial}_r\Gamma^{*s}_{h\ell}) - H^i_{skh}P^s_{r\ell}\right\}P^r_{jm}$$

$$= (\dot{\partial}_j a_{\ell m})H^i_{kh} + a_{\ell m}H^i_{jkh} + (\dot{\partial}_j b_{\ell m})(\delta^i_k y_h - \delta^i_h y_k)$$

$$+b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk})\ .$$

Using (*2.3*) in (*2.27*), we get

$$(2.28)\qquad \{H^r_{kh}(\dot{\partial}_j\Gamma^{*i}_{r\ell}) - H^i_{rh}(\dot{\partial}_j\Gamma^{*r}_{k\ell}) - H^i_{rk}(\dot{\partial}_j\Gamma^{*r}_{h\ell}) - H^i_{rkh}P^r_{j\ell}\}_{|m}$$

$$+H^r_{kh|\ell}(\dot{\partial}_j\Gamma^{*i}_{rm}) - H^i_{rh|\ell}(\dot{\partial}_j\Gamma^{*r}_{km}) - H^i_{rk|\ell}(\dot{\partial}_j\Gamma^{*r}_{hm}) - \ H^i_{kh|r}(\dot{\partial}_j\Gamma^{*r}_{m\ell})$$

$$-\left\{H^i_{rkh|\ell} + H^s_{kh}(\dot{\partial}_r\Gamma^{*i}_{s\ell}) - H^i_{sh}(\dot{\partial}_r\Gamma^{*s}_{k\ell}) - H^i_{sk}(\dot{\partial}_r\Gamma^{*s}_{h\ell}) - H^i_{skh}P^s_{r\ell}\right\}P^r_{jm}$$

$$= (\dot{\partial}_j a_{\ell m})H^i_{kh} + (\dot{\partial}_j b_{\ell m})(\delta^i_k y_h - \delta^i_h y_k).$$

Transvecting (*2.28*) by y^k , using (*1.1d*), (*1.1a*), (*1.3b*), (*1.4*) and (*1.6*), we get

$$(2.29)\qquad \{H^r_h(\dot{\partial}_j\Gamma^{*i}_{r\ell}) - H^i_r(\dot{\partial}_j\Gamma^{*r}_{h\ell}) - 2H^i_{rh}P^r_{j\ell}\}_{|m} + H^r_{h|\ell}(\dot{\partial}_j\Gamma^{*i}_{rm}) - H^i_{rh|\ell}(\mathrm{P}^r_{jm})$$

$$-H^{i}_{r|\ell}(\dot{\partial}_j \Gamma^{*r}_{hm}) - H^{i}_{h|r}(\dot{\partial}_j \Gamma^{*r}_{m\ell}) - \{H^{i}_{rh|\ell} + H^{s}_{h}(\dot{\partial}_r \Gamma^{*i}_{s\ell}) - H^{i}_{s}(\dot{\partial}_r \Gamma^{*s}_{h\ell})$$

$$-2H^{i}_{sh}P^{s}_{r\ell}\}P^{r}_{jm} = (\partial_j a_{\ell m})H^{i}_{h} + (\partial_j b_{\ell m})(y^i y_h - \delta^i_h F^2) .$$

Substituting the value of $\dot{\partial}_j b_{\ell m}$ from (*2. 14*) , in *(2. 29*), we get

(2.30) $$\{H^{r}_{h}(\dot{\partial}_j \Gamma^{*i}_{r\ell}) - H^{i}_{r}(\dot{\partial}_j \Gamma^{*r}_{h\ell}) - 2H^{i}_{rh}P^{r}_{j\ell}\}_{|m} + H^{r}_{h|\ell}(\dot{\partial}_j \Gamma^{*i}_{rm}) - H^{i}_{rh|\ell}(P^{r}_{jm})$$

$$-H^{i}_{r|\ell}(\dot{\partial}_j \Gamma^{*r}_{hm}) - H^{i}_{h|r}(\dot{\partial}_j \Gamma^{*r}_{m\ell}) - \{H^{i}_{rh|\ell} + H^{s}_{h}(\dot{\partial}_r \Gamma^{*i}_{s\ell}) - H^{i}_{s}(\dot{\partial}_r \Gamma^{*s}_{h\ell})$$

$$-2H^{i}_{sh}P^{s}_{r\ell}\}P^{r}_{jm} = (\dot{\partial}_j a_{\ell m})[H^{i}_{h} - H(\delta^i_h - l^i l_h)].$$

if

(2.31) $$\{H^{r}_{h}(\dot{\partial}_j \Gamma^{*i}_{r\ell}) - H^{i}_{r}(\dot{\partial}_j \Gamma^{*r}_{h\ell}) - 2H^{i}_{rh}P^{r}_{j\ell}\}_{|m} + H^{r}_{h|\ell}(\dot{\partial}_j \Gamma^{*i}_{rm}) - H^{i}_{rh|\ell}(P^{r}_{jm})$$

$$-H^{i}_{r|\ell}(\dot{\partial}_j \Gamma^{*r}_{hm}) - H^{i}_{h|r}(\dot{\partial}_j \Gamma^{*r}_{m\ell}) - \{H^{i}_{rh|\ell} + H^{s}_{h}(\dot{\partial}_r \Gamma^{*i}_{s\ell}) - H^{i}_{s}(\dot{\partial}_r \Gamma^{*s}_{h\ell}) - 2H^{i}_{sh}P^{s}_{r\ell}\}P^{r}_{jm} = 0 .$$

We have at least one of the following conditions :

(2.32) a) $(\dot{\partial}_j a_{\ell m}) = 0,$ b) $H^{i}_{h} = H(\delta^i_h - l^i l_h).$

Putting $H = F^2 R$, the equation (2. 32b) may be written as

(2.33) $$H^{i}_{h} = F^2 R(\delta^i_h - l^i l_h),$$

where $R \neq 0$. Therefore the space is a Finsler space of scalar curvature .

Thus , we have

Theorem 2.9. An $GH^h - BR - F_n$ for $n > 2$ admitting equation *(2.31)* holds is a Finsler space of scalar curvature provided $R \neq 0$, the covariant tensor $a_{\ell m}$ is not independent of directional arguments .

3. Conclusions

(3.1) The space whose defined by condition (2.3) is called generalized H^{h} − birecurrent Finsler space.

(3.2) Every generalized H^{h} − recurrent Finsler space is generalized H^{h} − birecurrent Finsler space**,** but the converse need not be true.

(3.3) In generalized $H^h -$ birecurrent Finsler space the Berwald curvature tensor H^i_{jkh} and the associate tensor H_{jrkh} satisfies the generalized birecurrence property .

(3.4) The torsion tensor H^i_{kh} , the deviation tensor H^i_h , the Ricci tensor H_{jk} , the vector H_k and the scalar curvature tensor H are all non- vanishing in our space .

(3.5) An generalized $H^h -$ birecurrent Finsler space is necessarily Landsberg space if and only if conditions (2.24), (2.19b) and $a_{j\ell} \neq \dot{\partial}_j a_\ell$ hold.

(3.6) An generalized $H^h -$ birecurrent Finsler space for $n > 2$ is a Finsler space of scalar curvature provided $R \neq 0$, the covariant tensor $a_{\ell m}$ is not independent of directional arguments and condition (2.31) holds .

4. Recommendations

The authors recommend that the research should be continued in the Finsler spaces because it has many applications in in relativity physics and other fields .

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