



Jungck Modified SP-Iterative Scheme

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Abstract

In this paper, we introduce Jungck-modified SP-iteration and we study its rate convergence with respect to the rate convergence Jungck-Mann, Jungck-Ishikawa, Jungck-Noor, Jungck-SP, Jungck-CR, Jungck-Karahan, Jungck-Picard S- iterative schemes.

Keywords: Jungck iterations; convergence; rate of convergence.

1. Introduction and Preliminaries

In 1976 Jungck has studied the common fixed points of self-mappings [1], but he has noted that Pfeffer [2] had discussed anointer dependence between the commuting mapping and the fixed point concepts, thus he highlights this interdependence but in a more general context.

1.1 Proposition [1]: Let S be a mapping on a set X into itself. Then S has a fixed point if and only if there is a constant mapping $T: X \rightarrow X$ which commutes with S i.e $T(S(x)) = S(T(x))$ for all x in X .

Hence he has extended this proposition to a common fixed point concept.

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1.2 Theorem [1]: Let S be a continuous mapping of a complete metric space (X, d) into itself. Then S has a fixed point in X if and only if there exists $L \in (0, 1)$ and a mapping $T: X \rightarrow X$ which commutes with S and satisfies

$$T(X) \subset S(X) \text{ and } d(T(x), T(y)) \leq kd(S(x), S(y)) \quad (1.1)$$

For all $x, y \in X$. Indeed S and T have common fixed point if (1.1) holds.

From the proof of the above theorem, the Jungck Picard iterative scheme is appeared

1.3 Definition [1]: Let C be a subset of a set X . $T, S: C \rightarrow C$ be two self mappings such that $T(C) \subseteq S(C)$ then for $x_0 \in C$

$$S(x_{n+1}) = T(x_n); n \in \mathbb{N}. \quad (1.2)$$

Also Jungck contraction mapping is introduced in [1].

1.4 Definition [1]: let C be a nonempty subset of a normed space X , then $T, S: C \rightarrow C$ are called Jungck-contraction mapping if there exist $k \in (0, 1)$ such that

$$\|Tx - Ty\| \leq k\|Sx - Sy\| \text{ for all } x, y \in C$$

Jungck- non expansive mapping if we have

$$\|Tx - Ty\| \leq \|Sx - Sy\| \text{ for all } x, y \in C$$

After that Jungck and his colleagues tried to weak the commuting condition. Thus in 1996 Jungck and his colleagues [3] introduced the following definition that dependence on confidence point concepts, recall that a point $x \in C$ is a coincidence point of S and T if $Sx = Tx$ [4], [5].

1.5 Definition [6]: Let (E, d) be a metric space, C be a nonempty subset of E , and S and T be self-maps of C . The pair (S, T) is called weakly compatible if $STx = TSx$ for all $x \in C(S, T)$.

Since 2005 many authors introduce different types of Jungck scheme like as the following.

1.6 Definition: Let C be a subset of a set X , $T, S: C \rightarrow C$ be two self mappings such that $T(C) \subseteq S(C)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[0, 1]$, then

1 The Jungck-Mann scheme: define the sequence $\{\hat{t}_n\}$ by Singh et al [7] as:

$$\begin{cases} \hat{t}_0 \in C \\ S\hat{t}_{n+1} = (1 - \alpha_n)S\hat{t}_n + \alpha_n T\hat{t}_n \end{cases} \quad (1.3)$$

2 The Jungck- Ishikawa scheme: define the sequence $\{S a_{n+1}\}$, by Olatinwo [8], as

$$\begin{cases} a_0 \in C \\ Sa_{n+1} = (1 - \alpha_n)Sa_n + \alpha_n Tb_n \\ Sb_n = (1 - \beta_n)Sa_n + \beta_n Ta_n; n \in \mathbb{N} \end{cases} \quad (1.4)$$

3 **The Jungck-Noor scheme:** define the sequence $\{Sw_n\}$, by [9] as

$$\begin{cases} w_0 \in C \\ Sw_{n+1} = (1 - \alpha_n)Sw_n + \alpha_n Tu_n \\ Su_n = (1 - \beta_n)Sw_n + \beta_n Tv_n \\ Sv_n = (1 - \gamma_n)Sw_n + \gamma_n Tw_n; n \in \mathbb{N} \end{cases} \quad (1.5)$$

4 **The Jungck-SP scheme:** define the sequence $S\hat{a}_n$, by Chugh and Kumar [10] as

$$\begin{cases} \hat{a}_0 \in C \\ S\hat{a}_{n+1} = (1 - \alpha_n)S\hat{b}_n + \alpha_n T\hat{b}_n \\ S\hat{b}_n = (1 - \beta_n)S\hat{c}_n + \beta_n T\hat{c}_n \\ S\hat{c}_n = (1 - \gamma_n)S\hat{a}_n + \gamma_n T\hat{a}_n; n \in \mathbb{N} \end{cases} \quad (1.6)$$

5 **The Jungck-CR scheme:** define the sequence $\{S\hat{w}_n\}$ [10] as

$$\begin{cases} \hat{w}_0 \in C \\ S\hat{w}_{n+1} = (1 - \alpha_n)S\hat{u}_n + \alpha_n T\hat{u}_n \\ S\hat{u}_n = (1 - \beta_n)S\hat{w}_n + \beta_n T\hat{v}_n \\ S\hat{v}_n = (1 - \gamma_n)S\hat{w}_n + \gamma_n T\hat{w}_n; n \in \mathbb{N} \end{cases} \quad (1.7)$$

Motivated by the above definition, we introduce the following Jungck type iterative procedures:

1.7 Definition: Let C be a subset of a set X , $T, S: C \rightarrow C$ be two self mappings such that $T(C) \subseteq S(C), \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[0,1]$

1 **The Jungck-Karahan scheme:** define the sequence $\{S\tilde{w}_n\}$ as

$$\begin{cases} \tilde{w}_0 \in C \\ S\tilde{w}_{n+1} = (1 - \alpha_n)T\tilde{w}_n + \alpha_n T\tilde{u}_n \\ S\hat{u}_n = (1 - \beta_n)S\tilde{w}_n + \beta_n T\tilde{v}_n \\ Sv_n = (1 - \gamma_n)S\tilde{w}_n + \gamma_n T\tilde{w}_n; n \in \mathbb{N} \end{cases} \quad (1.8)$$

2 **The Jungck-Picard-S scheme:** define the sequence $\{S\hat{x}_n\}$ as

$$\begin{cases} \hat{x}_0 \in C \\ S\hat{x}_n = T\hat{y}_n \\ S\hat{y}_n = (1 - \alpha_n)S\hat{x}_n + \alpha_n T\hat{z}_n \\ S\hat{z}_n = (1 - \beta_n)S\hat{x}_n + T\beta_n\hat{x}_n; n \in \mathbb{N} \end{cases} \quad (1.9)$$

3 **The Jungck-modified SP-iteration scheme:** define the sequence $\{Sx_n\}$ as

$$\begin{cases} x_0 \in C \\ Sx_n = Ty_n \\ Sy_n = (1 - \alpha_n)Sz_n + \alpha_n Tz_n \\ Sz_n = (1 - \beta_n)Sx_n + \beta_n Tx_n ; n \in \mathbb{N} \end{cases} \quad (1.10)$$

Where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[0,1]$.

In 2013, Aggarwal and his colleagues [11] introduce the concept of Jungcknon-expansive mapping

1.8 Defotion1.4 [11]: let C be a nonempty subset of normed linear space X , then $T, S: C \rightarrow C$ are called Jungck-nonexpansive mapping if we have

$$\|Tx - Ty\| \leq \|Sx - Sy\| \text{ for all } x, y \in C$$

1.9 Lemma [12]: Let X be uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequence in X such that $\lim_{n \rightarrow \infty} \sup x_n \leq a$, $\lim_{n \rightarrow \infty} \sup y_n \leq a$ and $\lim_{n \rightarrow \infty} \sup \|t_n x_n + (1 - t_n) y_n\| = a$, holds for some $a \geq 0$, then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

In the next three sections we establish our main results. In the second section, we will discuss the results of strong convergence and common fixed point of Jungck-modified SP-iteration. In the third section, compare speed of Jungck-modified SP-iteration with various Jungck-iterative shames with certain condition. In the fourth section, we prove the stability result of and established data dependence result of Jungck-modified SP-iteration.

2. Strong Convergence and Common Fixed Point

In this section we will discuss the strong convergence for the iterative (1.10).The following proportion states the images of the sequence $\{x_n\}$ under T, S are closed to each other as far as possible.

2.1 Proposition: Let C be a nonempty closed convex subset of a Banach space X , $T, S: C \rightarrow C$ be self-mappings satisfying Jungck- nonexpansive mappings. If $\{Sx_n\}$ generated by (1.10).with for some λ ; $0 \leq \lambda \leq \beta_n, \alpha_n \leq 1$ for all $n \in \mathbb{N}$, then:

- 1 $\lim_{n \rightarrow \infty} \|Sx_n - P\|$ exists for all $p \in F$
- 2 $\lim_{n \rightarrow \infty} \|Tx_n - Sx_n\| = 0$

Proof:part(1): Let p be a common fixed point of S and T , then:

$$\begin{aligned} \|Sx_{n+1} - P\| &= \|Ty_n - P\| \leq \|Sy_n - P\| = \|(1 - \alpha_n)Sz_n + \alpha_n Tz_n - P\| \\ &\leq (1 - \alpha_n)\|Sz_n - P\| + \alpha_n\|Tz_n - P\| \\ &\leq (1 - \alpha_n)\|Sz_n - P\| + \alpha_n\|Sz_n - P\| = \|Sz_n - P\| \end{aligned} \quad (2.1)$$

Now,

$$\|Sx_n - P\| = \|(1 - \beta_n)Sx_n + \beta_nTx_n - p\| \leq (1 - \beta_n)\|Sx_n - P\| + \beta_n\|Tx_n - p\| = \|Sx_n - p\| \quad (2.2)$$

From (2.1) and (2.2) we have, $\|Sx_{n+1} - P\| \leq \|Sx_n - p\|$ for all $n \in \mathbb{N}$. This shows that $\|Sx_n - p\|$ is non-increasing, thus the prove of part(1) is complete.

Part(2): Now we will prove that $\lim_{n \rightarrow \infty} \|Sx_n - P\| = 0$:

By part(1) there exists $c \in \mathbb{R}$; $\lim_{n \rightarrow \infty} \|Sx_n - P\| = c$ implies that

$$\lim_{n \rightarrow \infty} \sup \|Sx_n - P\| \leq c \quad (2.3)$$

Since $\|Tx_n - P\| \leq \|Sx_n - p\|$ implies that

$$\lim_{n \rightarrow \infty} \sup \|Tx_n - p\| \leq c \quad (2.4)$$

From (2.1) we get, $c \leq \lim_{n \rightarrow \infty} \inf \|Sx_n - p\|$

By (2.2) we get:

$$\lim_{n \rightarrow \infty} \sup \|Sx_n - p\| \leq c \quad (2.5)$$

Form (2.4) (2.5)we obtain that:

$$\lim_{n \rightarrow \infty} \sup \|Sx_n - p\| \leq c \leq \lim_{n \rightarrow \infty} \inf \|Sx_n - p\| \quad (2.6)$$

So $\lim_{n \rightarrow \infty} \|Sx_n - p\| = c$. Hence, this implies that

$$c = \lim_{n \rightarrow \infty} \|Sx_n - p\| = \lim_{n \rightarrow \infty} \|(1 - \beta_n)(Sx_n - p) + \beta_n(Tx_n - p)\| \quad (2.7)$$

By using lemma (1.3) and from (2.3), (2.4),(2.7) we get $\lim_{n \rightarrow \infty} \|Tx_n - Sx_n\| = 0$.

The following corollary show that $\{Sx_n\}$ defined by the three-stepiterative Jungck-modified SP-iterative (1.10) converge strongly to a fixed point of F with a certain conditions.

2.2 Corollary: Suppose that all hypotheses of proposition (2.1) are satisfied. If $\lim_{n \rightarrow \infty} \inf d(Sx_n, F) = 0$ or $\lim_{n \rightarrow \infty} \sup d(Sx_n, F) = 0$, then $\{Sx_n\}$ converge strongly to a fixed point of S and T .

Proof: By Proposition (2.11), we know $\lim_{n \rightarrow \infty} \|Sx_n - P\|$ exists for all $p \in F$, therefore $d(Sx_n, F)$ exists for all $p \in F$. But by hypothesis we get $\lim_{n \rightarrow \infty} \inf d(Sx_n, F) = \lim_{n \rightarrow \infty} \sup d(Sx_n, F) = 0$. Hence $d(Sx_n, F) = 0$. We will show that $\{Sx_n\}$ is a Cauchy sequence in C , let $\varepsilon > 0$. Since $d(Sx_n, F) = 0$, then there exists $n_0 \in \mathbb{N}$ such that

$d(Sx_n, F) < \frac{\varepsilon}{4}$ for all $n \geq n_0$. In particular, $\inf\{\|Sx_{n_0} - p\|\} \leq \frac{\varepsilon}{4}$. Hence there exists $p^* \in F$ such that $\|Sx_{n_0} - p^*\| \leq \frac{\varepsilon}{2}$. Now for all $m, n \leq n_0, \|Sx_{n+m} - Sx_n\| \leq \|Sx_{n+m} - p^*\| + \|Sx_n - p^*\| \leq \varepsilon$. Hence $\{Sx_n\}$

is Cauchy sequence, but C is closed in a Banach space, therefore it must be converge to a point z in C . Since $\lim_{n \rightarrow \infty} d(Sx_n, F) = 0$, gives that $z \in F$, thus $\{Sx_n\}$ Converge strongly to a fixed point $z \in F$. ■

The following corollary prove that $\{Sx_n\}$ defined by the three-step iterative Jungck-modified SP-iterative (1.10) converges strongly to a fixed point in F with different certain conditions.

2.3 Corollary: Suppose all hypothesis of (2.1) are satisfied. If there exists a non-decreasing function $f: [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > r$ for all $r \in (0, \infty)$ such that $f(d(Sx, F)) \leq \|Sx - Tx\|$ for all $x \in C$, then $\{Sx_n\}$ converge strongly to a point of F .

Proof: From Proposition(2.1), $\lim_{n \rightarrow \infty} \|Tx_n - Sx_n\| = 0$. thus, $\lim_{n \rightarrow \infty} f(d(Sx, F)) \leq \lim_{n \rightarrow \infty} \|Sx - Tx\| = 0$, that is, $\lim_{n \rightarrow \infty} f(d(Sx, F)) = 0$. Since $f: [0, \infty) \rightarrow [0, \infty)$ is non-decreasing function satisfying $f(0) = 0$, $f(r) > r$ for all $r \in (0, \infty)$, hence $\lim_{n \rightarrow \infty} d(Sx_n, F) = 0$. Thus the conditions of Corollary(2.2) are hold, therefore $\{Sx_n\}$ converges strongly to a point of F . ■

The following theorem show that Jungck-modified SP-iterative (1.10) convergence strongly to unique common fixed point of F .

2.4 Theorem: Let X be a Banach space, and $S, T: X \rightarrow X$ are Jungck-contraction mappings, weakly compatible assume that $T(X) \subseteq S(X)$, let $Sq = Tq = p$, and $\{Sx_n\}$ generated by (1.10) with some λ ; $0 \leq \lambda \leq \beta_n$, $\alpha_n \leq 1$. Then the Jungck modified SP-iteration converge strongly to p . Moreover p is the unique common fixed point of (S, T) .

Proof: Since S, T are Jungck contraction mappings then

$$\begin{aligned} \|Sx_{n+1} - p\| &= \|Ty_n - p\| \\ &\leq L\|Sy_n - p\| \\ &= L[1 - \alpha_n(1 - L)]\|Sz_n - p\| \\ &= L[1 - \lambda(1 - L)]\|Sz_n - p\| \end{aligned} \tag{2.8}$$

But,

$$\begin{aligned} \|Sz_n - p\| &= \|(1 - \beta_n)Sx_n + \beta_nTx_n - p\| \\ &\leq (1 - \beta_n)\|Sx_n - p\| + \beta_n\|Tx_n - p\| \\ &\leq (1 - \beta_n(1 - L))\|Sx_n - p\| \\ &\leq (1 - \lambda(1 - L))\|Sx_n - p\| \end{aligned} \tag{2.9}$$

Substituting (2.8) in (2.9) we have, $\|Sx_{n+1} - p\| \leq [L(1 - \lambda(1 - L))]^2 \|Sx_0 - p\|$. By indication we get

$$\|Sx_{n+1} - p\| \leq [L(1 - \lambda(1 - L))]^{2n} \|Sx_0 - p\|$$

Take the limit for both sides and since $(1 - \lambda(1 - L)) \in (0,1)$, thus $\{Sx_n\}$ converges strongly to p .

the uniqueness of the fixed point p of (S, T) comes from S, T are Jungck-contraction mappings and $0 < k < 1$.

Now since S, T are weakly compatible and $p = Tq = Sq$ then $Tp = TSq = STq$. Hence $Tp = Sp$ and $TTp = TSp = STp$, therefore Tp is coincidence of S, T but the coincidence point is unique, so $p = Tp$. Thus $Sp = Tp = p$. Therefore p is the unique common fixed point of (S, T) . ■

3. Rate of Converge

We turning our iteration to the speed of varies Jungck iterative schemes with Jungck-modified SP-iterative scheme for a pair of Jungck contraction mappings using certain condition. First we give a theorem that say Jungck-modified SP-iteration (1.10) converges faster than Jungck-Mann iteration (1.3), Jungck-Ishikawa iterative (1.4) to $p \in F$.

3.1 Theorem: Let X be a normed space and C be a nonempty closed convex subset of $X, S, T: C \rightarrow C$ be self mappings satisfied Jungck-contraction mappings, such that $T(C) \subseteq S(C)$. If there exist $\lambda; 0 \leq \lambda \leq \beta_n, \alpha_n \leq 1$ for all n , then the Jungck-modified SP-iteration (1.10) converges faster than Jungck-Mann(1.3) and Jungck-Ishikawa (1.4) to $p \in F$.

Proof: Let $p \in F$, for Jungck modified SP-iteration (1.10), we obtain

$$\begin{aligned} \|Sx_{n+1} - p\| &= \|Ty_n - p\| \leq L\|Sy_n - p\| \\ &\leq L[1 - \alpha_n(1 - L)]\|(1 - \beta_n)Sx_n + \beta_nTx_n - p\| \\ &\leq L[1 - \alpha_n(1 - L)][(1 - \beta_n(1 - L))\|Sx_n - p\| \\ &\quad \leq (1 - \lambda(1 - L))^2\|Sx_n - p\| \end{aligned}$$

By induction, we get, $\|Sx_{n+1} - p\| \leq [L(1 - \lambda(1 - L))]^n \|Sx_0 - p\|$

Put:

$$JMS_n := [L(1 - \lambda(1 - L))]^{2n} \|Sx_0 - p\| \quad (3.1)$$

From Jungck-Mann iteration (1.3), and by induction, we obtain that

$$\|S\hat{t}_{n+1} - p\| \leq [(1 - \lambda(1 - L))]^n \|S\hat{t}_0 - p\|$$

Put,

$$JMA_n := [(1 - \lambda(1 - L))]^n \|S\hat{t}_0 - p\|$$

From Jungck-Ishikawa process (1.4), we obtain that

$$\|Sa_{n+1} - p\| \leq [1 - \lambda(1 - L) - L\lambda^2(1 - L)]^n \|Sa_0 - p\|$$

Put,

$$JIS_n := [1 - \lambda(1 - L) - L\lambda^2(1 - L)]^n \|Sa_0 - p\|$$

Now after simple compute we get:

$$\frac{JMS_n}{JMA_n} = \frac{[k(1 - \lambda(1 - k))]^{2n} \|Sx_0 - p\|}{[1 - \lambda(1 - L)]^n \|S\hat{t}_0 - p\|} \text{ as } n \rightarrow \infty$$

and

$$\frac{JMS_n}{JIA_n} = \frac{[k(1 - \lambda(1 - k))]^{2n} \|Sx_0 - p\|}{[1 - \lambda(1 - L) - L\lambda^2(1 - L)]^n \|Sa_0 - p\|} \text{ as } n \rightarrow \infty$$

Hence $\{Sx_n\}$ converges to p faster than $\{S\hat{t}_n\}$ and $\{Sa_n\}$. ■

The following theorem shows that Jungck-modified SP-iterative (1.10) converges faster than Jungck-Noor (1.5) and Jungck-Karahan (1.8) to $p \in F$.

3.2 Theorem: Let C be a nonempty closed convex subset of a normed space $X, S, T: C \rightarrow C$ be self mappings satisfied Jungck-contraction mappings, assume that $T(C) \subseteq S(C)$. If there exist $\lambda; 0 \leq \lambda \leq \beta_n, \alpha_n \leq 1$ for all $n \in \mathbb{N}$, then the Jungck-Modified SP-iteration (1.10) converges faster than Jungck-Noor(1.5) and Jungck-Karahan (1.8) to $p \in F$.

Proof: Let $p \in F$, from Jungck-Karahan iterative (1.8), we get

$$\begin{aligned} \|S\tilde{w}_{n+1} - p\| &= \|(1 - \alpha_n)T\tilde{w}_n + \alpha_n T\tilde{u}_n - p\| \\ &\leq L(1 - \alpha_n)\|S\tilde{w}_n - p\| + \alpha_n L\|S\tilde{u}_n - p\| \\ &\leq L[(1 - \alpha_n)\|S\tilde{w}_n - p\| \\ &\quad + \alpha_n(L(1 - \beta_n)\|S\tilde{w}_n - p\| + L\beta_n(1 - \gamma_n)\|S\tilde{w}_n - p\| + L^2\beta_n\gamma_n\|S\tilde{w}_n - p\|)] \\ &\leq L(1 - \alpha_n(1 - L) - L\alpha_n\beta_n\gamma_n(1 - L))\|S\tilde{w}_n - p\| \\ &\leq L(1 - \lambda(1 - L) - L\lambda^3(1 - L))\|S\tilde{w}_n - p\| \\ &\quad \vdots \\ &\leq [L(1 - \lambda(1 - L) - L\lambda^3(1 - L))]^n \|S\tilde{w}_0 - p\| \end{aligned}$$

Let,

$$JKA_n := [L(1 - \lambda(1 - L) - L\lambda^3(1 - L))]^n \|S\tilde{w}_0 - p\|$$

From Jungck-Noor iterative (1.5), we obtain that

$$\begin{aligned} \|Sw_{n+1} - p\| &= \|(1 - \alpha_n)Sw_n + \alpha_nTu_n - p\| \\ &\leq (1 - \alpha_n)\|Sw_n - p\| + L\alpha_n\|Su_n - p\| \\ &\leq (1 - \alpha_n)\|Sw - p\| + L\alpha_n(1 - \beta_n)\|Sw_n - p\| + L^2\alpha_n\beta_n\|Sv_n - p\| \\ &= (1 - \alpha_n)\|Sw_n - p\| + L\alpha_n(1 - \beta_n)\|Sw_n - p\| + L^2\alpha_n\beta_n\|(1 - \gamma_n)Sw_n + \gamma_nTw_n - p\| \\ &\leq [1 - \lambda(1 - L) - L\lambda^2(1 - L) - L^2\lambda^3]\|Sw_n - p\| \\ &\vdots \\ &\leq [(1 - \lambda(1 - L) - L\lambda^2(1 - L) - L^2\lambda^3(1 - L))]^n\|Sw_0 - p\| \\ JNO_n &:= [1 - \lambda(1 - L) - L\lambda^2(1 - L) - L^2\lambda^3(1 - L)]^n\|Sw_0 - p\| \end{aligned}$$

From Jungck-modified SP-iteration we have

$$JMS_n := [L(1 - \lambda(1 - L))]^{2n}\|Sx_0 - p\|$$

Now since:

$$\begin{aligned} \frac{JMS_n}{JKAn} &= \frac{[L(1 - \lambda(1 - L))]^{2n}\|Sx_0 - p\|}{[L(1 - \lambda(1 - L) - L\lambda^2(1 - L) - L^2\lambda^3(1 - L))]^n\|S\tilde{w}_0 - p\|} \text{ as } n \rightarrow \infty \\ \frac{JMS_n}{JNO_n} &= \frac{[L(1 - \lambda(1 - L))]^{2n}\|Sx_0 - p\|}{[1 - \lambda(1 - L) - L\lambda^2(1 - L) - L^2\lambda^3(1 - L)]^n\|Sw_0 - p\|} \text{ as } n \rightarrow \infty \end{aligned}$$

Hence $\{Sx_n\}$ converges to p faster than $\{S\tilde{w}_n\}$ and $\{Sw_n\}$. ■

The next theorem prove that Jungck-modified SP-iterative (1.10) convergence faster than Jungck-CR (1.7) and Jungck-SP (1.9) to $p \in F$.

3.3 Theorem: Let C be a nonempty closed convex subset of a normed space $X, S, T: C \rightarrow C$ be self mappings satisfied Jungck-contraction mappings, assume that $T(C) \subseteq S(C)$. If there exist $\lambda; 0 \leq \lambda \leq \beta_n, \alpha_n \leq 1$ for all $n \in \mathbb{N}$, then the Jungck-Modified SP-iteration (1.10) converges faster than Jungck-CR (1.7) and SP-iteration(1.6) to $p \in F$.

Proof: Let $p \in F$ from JCR (1.7) iterative, we obtain that

$$\begin{aligned} \|S\hat{w}_{n+1} - p\| &= \|(1 - \alpha_n)S\hat{u}_n + \alpha_nT\hat{u}_n - p\| \\ &\leq (1 - \alpha_n)\|S\hat{u}_n - p\| + \alpha_nL\|S\hat{u}_n - p\| \\ &\leq (1 - \alpha_n(1 - L))\|S\hat{u}_n - p\| \\ &= L(\alpha_n(1 - L) [L(1 - \beta_n)\|S\hat{w}_n - p\| + L\beta_n(\|(1 - \gamma_n)S\hat{w}_n + \gamma_nT\hat{w}_n - p\|)]) \\ &\leq L(1 - \alpha_n(1 - L))(1 - L\beta_n\gamma_n(1 - L))\|S\hat{w}_n - p\| \\ &\leq L(1 - \lambda(1 - L))(1 - \lambda^2(1 - L))\|S\hat{w}_n - p\| \\ &\vdots \\ &\leq [L(1 - \lambda(1 - L))(1 - \lambda^2(1 - L))]^n\|S\hat{w}_0 - p\| \end{aligned}$$

Let,

$$JCR_n := [L(1 - \lambda(1 - L))(1 - \lambda^2(1 - L))]^n \|S\hat{w}_0 - p\|$$

From Jungck-SP-iteration (1.9), we obtain that

$$\begin{aligned} \|S\hat{a}_{n+1} - p\| &= \|(1 - \alpha_n)S\hat{b}_n + \alpha_n T\hat{b}_n - p\| \\ &\leq (1 - \alpha_n(1 - L))\|S\hat{b}_n - p\| \\ &= (1 - \alpha_n(1 - L))[(1 - \beta_n)\|S\hat{c}_n - p\| + L\beta_n\|Sc_n - p\|] \\ &\leq (1 - \alpha_n(1 - L))(1 - \beta_n(1 - L))\|S\hat{c}_n - p\| \\ &\leq (1 - \alpha_n(1 - L))(1 - \beta_n(1 - L))(1 - \gamma_n(1 - L))\|S\hat{a}_n - p\| \\ &\leq ((1 - \lambda(1 - L))^3)\|S\hat{a}_n - p\| \\ &\vdots \\ &\leq [(1 - \lambda(1 - L))]^{3n}\|S\hat{a}_0 - p\| \end{aligned}$$

Put,

$$JSP_n := [(1 - \lambda(1 - L))]^{3n}\|S\hat{a}_0 - p\|$$

Finally, we get from Jungck-modified SP-iteration

$$JMS_n := [L(1 - \lambda(1 - L))]^{2n}\|Sx_0 - p\|$$

Now since:

$$\begin{aligned} \frac{JMS_n}{JSP_n} &= \frac{[L(1 - \lambda(1 - L))]^{2n}\|Sx_0 - p\|}{[(1 - \lambda(1 - L))]^{3n}\|S\hat{a}_0 - p\|} \text{ as } n \rightarrow \infty \\ \frac{JMS_n}{JCR_n} &= \frac{[L(1 - \lambda(1 - L))]^{2n}\|Sx_0 - p\|}{[L(1 - \lambda(1 - L))(1 - \lambda^2(1 - L))]^n\|S\hat{w}_0 - p\|} \text{ as } n \rightarrow \infty \end{aligned}$$

Hence $\{Sx_n\}$ converges to p faster than $\{S\hat{w}_0\}$ and $\{S\hat{a}_0\}$. ■

3.4 Theorem: Let C be a nonempty closed convex subset of a normed space X , $S, T: C \rightarrow C$ be self mappings satisfied Jungck-contraction mappings, assume that $T(C) \subseteq S(C)$. If there exist $\lambda; 0 \leq \lambda \leq \beta_n, \alpha_n \leq 1$ for all $n \in \mathbb{N}$, then the Jungck-Picard-S (1.9) converge faster than Jungck-modified SP-iteration (1.10) to $p \in F$.

Proof: Let $p \in F$, for Jungck modified SP-iteration (1.10), we obtain

$$\begin{aligned} \|S\hat{x}_{n+1} - p\| &= \|T\hat{y}_n - p\| \\ &\leq L\|S\hat{y}_n - p\| \\ &= L[(1 - \alpha_n)\|T\hat{x}_n - p\| + \alpha_n\|T\hat{z}_n - p\|] \\ &\leq L^2[(1 - \alpha_n)\|T\hat{x}_n - p\| + \alpha_n\|S\hat{z}_n - p\|] \end{aligned}$$

$$\begin{aligned}
 &= L^2[(1 - \alpha_n)\|S\hat{x}_n - p\| + \alpha_n\|(1 - \beta_n) S\hat{x}_n + \beta_n T\hat{x}_n - p\|] \\
 &\leq L^2[(1 - \alpha_n)\|S\hat{x}_n - p\| + \alpha_n(1 - \beta_n)\|S\hat{x}_n - p\| + L\alpha_n\beta_n\|S\hat{x}_n - p\|] \\
 &= L^2[((1 - \alpha_n) + \alpha_n(1 - \beta_n) + L\alpha_n\beta_n)\|S\hat{x}_n - p\|] \\
 &\leq L^2(1 - \lambda^2(1 - L))\|S\hat{x}_n - p\| \\
 &\vdots \\
 &\leq [L^2(1 - \lambda^2(1 - L))]^n\|S\hat{x}_0 - p\|
 \end{aligned}$$

Put,

$$JPS_n := [L^2(1 - \lambda^2(1 - L))]^n\|S\hat{x}_0 - p\|$$

Thus we get from Jungck-modified SP-iteration

$$JMS_n := [L(1 - \lambda(1 - L))]^{2n}\|Sx_0 - p\|$$

Now since:

$$\frac{JPS_n}{JMS_n} = \frac{[L^2(1 - \lambda^2(1 - L))]^n\|S\hat{x}_0 - p\|}{[L(1 - \lambda(1 - L))]^{2n}\|Sx_0 - p\|} \text{ as } n \rightarrow \infty$$

Hence $\{S\hat{x}_n\}$ converges to p faster than $\{Sx_n\}$. ■

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