



Killing Vector Fields on the Lorentzian Manifold of Non-Static Special Axially Symmetric Space-Time

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Abstract

In this work direct integration and algebraic techniques have been implied to study the Killing vector fields in special non-static axially symmetric space-time. It has been shown that different classes of above space-time admit Killing vector fields of different dimensions. The dimensions of Killing Lie algebras are 1,2,3,6 and 10.

Keywords: Killing vector fields; direct integration technique; Lie algebra.

1. Introduction

The main objective of this work is to discuss the Killing symmetry in special non-static axially symmetric space-time by using direct integration technique. Here four dimensional, connected, Hausdorff space-time manifold with Lorentz metric g of signature $(-, +, +, +)$ is denoted by M . The curvature tensor and the Ricci tensor associated with g_{ab} , through the Levi-Civita connection are denoted in component form by R^a_{bcd} and $R_{ab} = R^c_{acb}$ respectively. The symbol L , semicolon and comma denote the Lie, covariant and partial derivatives respectively. Symmetrization and skew-symmetrization are denoted by round and square brackets, respectively.

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On the manifold M , the covariant derivative of any vector field X can be split as [1],

$$X_{a;b} = \frac{1}{2}h_{ab} + G_{ab}, \tag{1.1}$$

where $G_{ab} (= -G_{ba})$ is a skew symmetric and $h_{ab} (= h_{ba}) = L_X g_{ab}$ is a symmetric tensor on the manifold M , respectively. If $h_{ab} = 2\alpha g_{ab}$, $\alpha \in R$

$$\text{Equivalently, } g_{ab,c}X^c + g_{bc,a}X^c + g_{ac,b}X^c = 2\alpha g_{ab} \tag{1.2}$$

If $\alpha \neq 0$ then X is said to be homothetic (and becomes Killing if $\alpha = 0$). The vector field X is said to be proper homothetic if it is not killing vector field [2, 3]. In terms of Lie brackets, the Lie algebra of a set of vector fields on a manifold is completely characterized by the structure constant C^a_{bc} given by

$$[X_b, X_c] = C^a_{bc} X_a, \quad C^a_{bc} = -C^a_{cb}, \tag{1.3}$$

Where X_a shows the generator. In recent years much interest has been shown to study different symmetries by many authors Bokhari and his colleagues [4], explored killing vectors of spherically symmetric static space-time. Amjad Ali and his colleagues [5-10], worked on Killing, homothetic and conformal symmetries. Tooba Feroz and his colleagues [11] investigated killing vector fields of non-static plane symmetric space-time. A. Qadir and his colleagues [12-14], worked on static cylindrically and spherical symmetries. In [15, 16], M. Ziad, and his colleagues worked on Homothetic motions of spherically symmetric and classification of static plane symmetric space-time. In [17, 18], Ghulam Shabir worked on Classification of cylindrically symmetric static space time according to their proper homothetic vector fields.

2. Main Results

Consider a special non-static axially symmetric space-time in the usual cylindrical coordinate system

(t, r, θ, ϕ) (labeled by (x^0, x^1, x^2, x^3) , respectively) with line element [19],

$$ds^2 = -e^{A(t,r,\theta)} dt^2 + e^{B(t,r,\theta)} (dr^2 + d\theta^2 + d\phi^2), \tag{2.1}$$

The above space-time admits only one Killing vector field which is $\frac{\partial}{\partial \phi}$.

A vector field X is said to be Killing vector field if it satisfies the equation $L_X g_{ab} = 0$.

$$\text{Equivalently, } g_{ab,c} X^c + g_{bc} X^c_{,a} + g_{ac} X^c_{,b} = 0. \tag{2.2}$$

One can write (2.2) explicitly as

$$A_t(t,r,\theta) X^0 + A_r(t,r,\theta) X^1 + A_\theta(t,r,\theta) X^2 + 2X^0_{,0} = 0 \tag{2.3}$$

$$e^{A(t,r,\theta)} X^0_{,1} - e^{B(t,r,\theta)} X^1_{,0} = 0 \tag{2.4}$$

$$e^{A(t,r,\theta)} X^0_{,2} - e^{B(t,r,\theta)} X^2_{,0} = 0 \tag{2.5}$$

$$e^{A(t,r,\theta)} X^0_{,3} - e^{B(t,r,\theta)} X^3_{,0} = 0 \tag{2.6}$$

$$B_t(t,r,\theta) X^0 + B_r(t,r,\theta) X^1 + B_\theta(t,r,\theta) X^2 + 2X^1_{,1} = 0 \tag{2.7}$$

$$X^2_{,1} + X^1_{,2} = 0 \tag{2.8}$$

$$X^3_{,1} + X^1_{,3} = 0 \tag{2.9}$$

$$B_t(t,r,\theta) X^0 + B_r(t,r,\theta) X^1 + B_\theta(t,r,\theta) X^2 + 2X^2_{,2} = 0 \tag{2.10}$$

$$X^3_{,2} + X^2_{,3} = 0 \tag{2.11}$$

$$B_t(t,r,\theta) X^0 + B_r(t,r,\theta) X^1 + B_\theta(t,r,\theta) X^2 + 2X^3_{,3} = 0 \tag{2.12}$$

Solving equations (8), (12), (13) and (15) respectively, we have the following system

$$\left. \begin{aligned} X^0 &= e^{B(t,r,\theta)-A(t,r,\theta)} \int [E^3(t,r,\phi) + E^4(t,\theta,\phi)] d\phi + E^7(t,r,\theta) \\ X^1 &= E^5(t,r,\theta) + E^6(t,r,\phi) \\ X^2 &= E^1(t,r,\theta) + E^2(t,\theta,\phi) \end{aligned} \right\} \tag{2.13}$$

$$X^3 = E^3(t,r,\phi) + E^4(t,\theta,\phi)$$

Wher $E^1(t,r,\theta), E^2(t,\theta,\phi), E^3(t,r,\phi), E^4(t,\theta,\phi), E^5(t,r,\theta), E^6(t,r,\phi)$ and $E^7(t,\theta,\phi)$ are integrating functions of X^0, X^1, X^2, X^3 . These unknown functions will be determined by the use of remaining six equations. To avoid lengthy calculation we only present the results.

In the following we will discuss different possibilities when the space-time admits different number of Killing vector fields.

2.1 When there is only one killing symmetry

Case (1) In this case $A_t = 0, B_r = 0, B_\theta = 0$ and the space - time (2.1) takes the form

$$ds^2 = -e^{A(r,\theta)} dt^2 + e^{B(t)} (dr^2 + d\theta^2 + d\phi^2), \quad (2.14)$$

and in this case the Killing vector field is

$$X^0 = 0, X^1 = 0, X^2 = 0, X^3 = c_6. \quad (2.15)$$

Case (2) In this case $A_t = 0, B_\theta = 0$ and the space-time (2.1) reduced to

$$ds^2 = -e^{A(r,\theta)} dt^2 + e^{B(t,r)} (dr^2 + d\theta^2 + d\phi^2), \quad (2.16)$$

and in this case the Killing vector field is

$$X^0 = 0, X^1 = 0, X^2 = 0, X^3 = c_7. \quad (2.17)$$

Case (3) In this case $A_t = 0, B_r = 0$ and the space-time (2.1) reduced to

$$ds^2 = -e^{A(r,\theta)} dt^2 + e^{B(t,\theta)} (dr^2 + d\theta^2 + d\phi^2), \quad (2.18)$$

and in this case the Killing vector field is

$$X^0 = 0, X^1 = 0, X^2 = 0, X^3 = c_7. \quad (2.19)$$

2.2 When there are only two killing symmetries

Case (4) In this case $A_t = 0, B_r = 0, B_t = 0$ and the space-time (2.1) reduced to

$$ds^2 = -e^{A(r,\theta)} dt^2 + e^{B(\theta)} (dr^2 + d\theta^2 + d\phi^2), \quad (2.20)$$

and in this case the Killing vector field is

$$X^0 = c_9, X^1 = 0, X^2 = 0, X^3 = c_7. \quad (2.21)$$

Case (5) In this case $A_t = 0, B_t = 0$ and the space-time (2.1) reduced to

$$ds^2 = -e^{A(r,\theta)} dt^2 + e^{B(r,\theta)} (dr^2 + d\theta^2 + d\phi^2), \quad (2.22)$$

and in this case the Killing vector field is

$$X^0 = c_8, X^1 = 0, X^2 = 0, X^3 = c_7. \quad (2.23)$$

Case (6) In this case $A_t = 0, B_\theta = 0, B_t = 0$ and the space-time (2.1) becomes

$$ds^2 = -e^{A(r,\theta)} dt^2 + e^{B(r)} (dr^2 + d\theta^2 + d\phi^2), \quad (2.24)$$

and in this case the Killing vector field is

$$X^0 = c_{10}, X^1 = 0, X^2 = 0, X^3 = c_6. \quad (2.25)$$

Case (7) In this case $A_t = 0, B(t, r, \theta) = c$ (constant) and the space-time (2.1) takes the form

$$ds^2 = -e^{A(r,\theta)} dt^2 + e^c (dr^2 + d\theta^2 + d\phi^2), \quad (2.26)$$

and in this case the Killing vector field is

$$X^0 = c_7, X^1 = 0, X^2 = 0, X^3 = c_3 \quad (2.27)$$

2.3 When there are only three killing symmetries

Case (8) In this case $A_r = 0, B_r = 0, B_\theta = 0,$ and the space-time (2.1) takes the form

$$ds^2 = -e^{A(t,\theta)} dt^2 + e^{B(t)} (dr^2 + d\theta^2 + d\phi^2), \quad (2.28)$$

and in this case the Killing vector field is

$$X^0 = 0, X^1 = -c_3\phi + c_9, X^2 = 0, X^3 = c_3r + c_7 \quad (2.29)$$

Case (9) In this case $A_r = 0, B(t, r, \theta) = c,$ and the space-time (2.1) takes the form

$$ds^2 = -e^{A(t,\theta)} dt^2 + e^c (dr^2 + d\theta^2 + d\phi^2), \quad (2.30)$$

and in this case the Killing vector field is

$$X^0 = 0, X^1 = -c_1\phi + c_4, X^2 = 0, X^3 = c_1r + c_3 \quad (2.31)$$

Case (10) In this case $A_\theta = 0, B_r = 0, B_\theta = 0,$ and the space-time (2.1) takes the form

$$ds^2 = -e^{A(t,r)} dt^2 + e^{B(t)} (dr^2 + d\theta^2 + d\phi^2), \tag{2.32}$$

and in this case the Killing vector field is

$$X^0 = 0, X^1 = 0, X^2 = c_6\phi + c_8, X^3 = -c_6\theta + c_7 \tag{2.33}$$

2.4 When there are only six killing symmetries

Case (11) In this case $A_r = 0, A_\theta = 0, B(t, r, \theta) = c$, (constant) and the space-time (2.1) takes the form

$$ds^2 = -e^{A(t)} dt^2 + e^c (dr^2 + d\theta^2 + d\phi^2), \tag{2.34}$$

and in this case the Killing vector field is

$$\left. \begin{aligned} X^0 &= 0, X^1 = -c_3\theta - c_1\phi + c_5, \\ X^2 &= c_3r + c_2\phi + c_6, X^3 = c_1r - c_2\theta + c_4 \end{aligned} \right\} \tag{2.35}$$

2.5 When there are only ten killing symmetries

Case (12) In this case $A(t, r, \theta) = c_1$ (constant), $B(t, r, \theta) = c_2$ (constant), and the space-time (2.1) will become

Minkowski and takes the form

$$ds^2 = -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{2.36}$$

and in this case the Killing vector field is

$$\left. \begin{aligned} X^0 &= e^{c_2-c_1} [c_{12}r + c_7\phi] + c_{14}, \\ X^1 &= c_{12}t - c_9\theta - c_6\phi + c_{13}, \\ X^2 &= c_{10}t + c_9r + c_4\phi + c_{11}, \\ X^3 &= c_7t + c_6r - c_4\theta + c_8 \end{aligned} \right\} \tag{2.37}$$

3. Conclusion

In this paper a study of Killing symmetry is discussed to study the non-static axially symmetric space-time. An approach is adopted to study the above space-time is direct integration technique. Some special cases are discussed and it has been shown that the Killing Lie algebra are of dimensions 1, 2, 3, 6, and 10.

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