



Numerical Simulation of One-Dimensional Shallow Water Equations

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Abstract

In this study, a relatively new semi-analytic technique, the reduced differential transform method is employed to obtain high accurate solutions of the famous coupled partial differential equations with physical interests namely the variable-depth shallow water equations with source term. The solutions are calculated in the form of a convergent power series with easily computable components. The Reduced differential transform method is easy to apply, reduces the size of computations, and produces an approximate solution without any discretization or perturbation. The results show the accuracy and efficiency of the reduced differential transform method in comparison to other existing methods.

Keywords: Reduced differential transform method; shallow water equations; Conservation laws; Soliton solution; Error analysis.

1. Introduction

The notion of conservation laws plays an important role in the study of differential equations which are of great importance in many areas of physics. They are essential since they allow us to draw conclusions of a physical system under study in an efficient way. The mathematical idea of conservation laws comes from the formulation of familiar physical laws of conservation of energy, conservation of momentum and so on.

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The flow of water in a wide frictionless channel with rectangular cross-section and smoothly varying bottom surface is governed by system of conservation laws namely, the shallow water equations [1]. In the one-dimensional case, the flow of a fluid in an infinitely wide channel can be represented by [2]

$$\begin{aligned} u_t + (uv)_x &= 0 \\ v_t + (u + 0.5v^2)_x &= H'(x) \end{aligned}, \quad x \in \mathbb{R}, t > 0. \tag{1}$$

The functions $u(x, t)$ and $v(x, t)$ represent the total height above the bottom of the channel and the fluid velocity, respectively, and $H(x)$ is the depth of a point from a fixed reference level of the water. The two independent variables x and t are the distance along the direction of flow and the time, respectively.

In 1994, Bermudez and Vazquez [3] found that no exact analytical solution for the shallow water equations when the bottom is not flat. But, using the theory of compensated compactness, Bruno [4] proves the convergence of a weak solution for one-dimensional non-homogeneous, non-strictly quasi-linear hyperbolic system. In order to solve the system in Eq.(1), many numerical methods were applied to solve the shallow water system. for example, the finite difference method [5], the finite element method [6], the Adomian decomposition method [2] and the variational iteration method [7].

In this paper, the relatively new technique, called the reduced differential transform method [8-9], is applied to solve the system in Eq. (1).

This paper is organized as follows: In section two, we begin with some basic definitions and theorems of the reduced differential transform method. Analytic solutions of the system of shallow water equations via mentioned method are given in section three. To show the effectiveness and convergence of proposed method, numerical experiment is considered in section four. Section five contains the conclusions and discussions.

2. Reduced Differential Transform Method

Consider the analytic and continuously differentiated function of two variables $u(x, t)$ and suppose that it can be represented as a product of two single-variable functions, i.e., $u(x, t) = f(x)g(t)$. Based on the properties of differential transform, the function $u(x, t)$ can be represented as

$$u(x, t) = \sum_{i=0}^{\infty} F(i) x^i \sum_{j=0}^{\infty} G(j) t^j = \sum_{k=0}^{\infty} U_k(x) t^k, \tag{2}$$

where $U_k(x)$ is the transformed function, called t -dimensional spectrum function of $u(x, t)$, and defined by

$$U_k(x) = \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x, t) \right]_{t=0} \tag{3}$$

The differential inverse transform of $U_k(x)$ is defined by combining Eqs.(2) and (3) implies that

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x,t) \right]_{t=0} t^k. \tag{4}$$

One can easily obtained that the reduced differential transform is derived from the power series expansion. Next, some basic theorems and generalized formulas of reduced differential transform [8-16] are listed.

Theorem 2.1. The reduced differential transform is linear.

Theorem 2.2. If $u(x,t) = x^m t^n$ then,

$$U_k(x) = x^m \delta(k-n), \quad \delta(k) = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}. \tag{5}$$

Theorem 2.3. If $u(x,t) = x^m t^n v(x,t)$ then,

$$U_k(x) = x^m V_{k-n}(x). \tag{6}$$

Theorem 2.4. If $u(x,t) = \frac{\partial^r}{\partial t^r} v(x,t)$ then,

$$U_k(x) = (k+1) \dots (k+r) V_{k+r}(x). \tag{7}$$

Theorem 2.5. If $u(x,t) = \frac{\partial^r}{\partial x^r} v(x,t)$ then,

$$U_k(x) = \frac{\partial^r}{\partial x^r} V_k(x). \tag{8}$$

We [10, 15] proved the following generalized reduced differential transforms:

Theorem 2.6. If $u(x,t) = v^n(x,t)$, $n \in \mathbb{N}$, then,

$$U_k(x) = \sum_{r_1=0}^k \sum_{r_2=0}^{k-r_1} \dots \sum_{r_{n-1}=0}^{k-\sum_{i=1}^{n-2} r_i} V_{r_1}(x) V_{r_2}(x) \dots V_{k-\sum_{i=1}^{n-1} r_i}(x). \tag{9}$$

Theorem 2.7. If $u(x,t) = v^{n-m}(x,t) \left(\frac{\partial}{\partial x} v(x,t) \right)^m$, $n, m \in \mathbb{N}$, then,

$$U_k(x) = \sum_{r_1=0}^k \sum_{r_2=0}^{k-r_1} \cdots \sum_{r_{n-1}=0}^{k-\sum_{i=1}^{n-2} r_i} V_{r_1}(x) V_{r_2}(x) \cdots V_{r_{n-1}}(x) \frac{\partial}{\partial x} V_{r_{n-1}} \cdots (x) V_{k-\sum_{i=1}^{n-1} r_i}(x). \quad (10)$$

Theorem 2.8. If $u(x, t) = \frac{\partial^{n+m}}{\partial x^n \partial t^m} v(x, t)$ then,

$$U_k(x) = (k+1) \dots (k+m) \frac{\partial^n}{\partial x^n} V_{k+m}(x). \quad (11)$$

3. Analytic Solutions for System of Shallow Water Equations

In this section, we apply the mentioned procedure in previous section for our problem subject to given initial data

$$\begin{aligned} u(x, 0) &= f(x) \\ v(x, 0) &= g(x) \end{aligned}, \quad x \in \mathbb{R}. \quad (12)$$

Operating the reduced differential transform to Eq. (1) and using related theorems, we get

$$\begin{aligned} U_{k+1}(x) &= \frac{1}{(k+1)} \left(-\sum_{r=0}^k U_r(x) \frac{\partial}{\partial x} V_{k-r}(x) - \sum_{r=0}^k V_r(x) \frac{\partial}{\partial x} U_{k-r}(x) \right), \\ V_{k+1}(x) &= H'(x) - \frac{1}{(k+1)} \left(\frac{\partial}{\partial x} U_k(x) + \sum_{r=0}^k U_r(x) \frac{\partial}{\partial x} V_{k-r}(x) \right). \end{aligned} \quad (13)$$

With converted starting values

$$\begin{aligned} U_0(x) &= f(x), \\ V_0(x) &= g(x). \end{aligned} \quad (14)$$

Following the reduced differential transform method, the nth order approximate solutions $u(x, t)$ and $v(x, t)$ are given by

$$\begin{aligned} u_n(x, t) &= \sum_{k=0}^n U_k(x) t^k, \\ v_n(x, t) &= \sum_{k=0}^n V_k(x) t^k. \end{aligned} \quad (15)$$

The exact solutions are

$$\begin{aligned} u(x, t) &= \lim_{n \rightarrow \infty} u_n(x, t), \\ v(x, t) &= \lim_{n \rightarrow \infty} v_n(x, t). \end{aligned} \tag{16}$$

Provided the series solutions converge and have closed forms.

4. Numerical Experiment

To demonstrate the convergence of the RDTM, the results of numerical example are presented in this section, and only few terms are required to obtain accurate solutions. Consider the system in Eq. (1) subject to initial conditions

$$\begin{aligned} u(x, 0) &= H(x) + 0.25 \operatorname{Sech}(x), \\ v(x, 0) &= 0 \end{aligned}, \quad x \in \mathbb{R}, \tag{17}$$

where

$$H(x) = \frac{e^{-x^2}}{1 + e^{-x^2}}. \tag{18}$$

Substituting given data into recurrence relations Eq. (13), the first 15 values of $U_k(x)$ and $V_k(x)$ were computed recursively with aid of Mathematica. The approximate series solutions of height and velocity of the water, respectively, as

$$\begin{aligned} u(x, t) &= \sum_{k=0}^{15} U_k(x) t^k, \\ v(x, t) &= \sum_{k=0}^{15} V_k(x) t^k. \end{aligned} \tag{19}$$

which are represented in Figure 1.

In this example, we cannot determine the errors in comparative to the exact solutions since we do not know these solutions. Instead, we define absolute errors to obtain the accuracy of used scheme, by substituting obtained approximate solutions into our system in Eq. (1) and comparing to zeros right hand sides. That is

$$\begin{aligned} E_{abs}(u) &= \left| u_t + (uv)_x \right|, \\ E_{abs}(v) &= \left| v_t + (u + 0.5v^2)_x - H'(x) \right|. \end{aligned} \tag{20}$$

The absolute errors related to approximate solutions are shown in Figure 2.

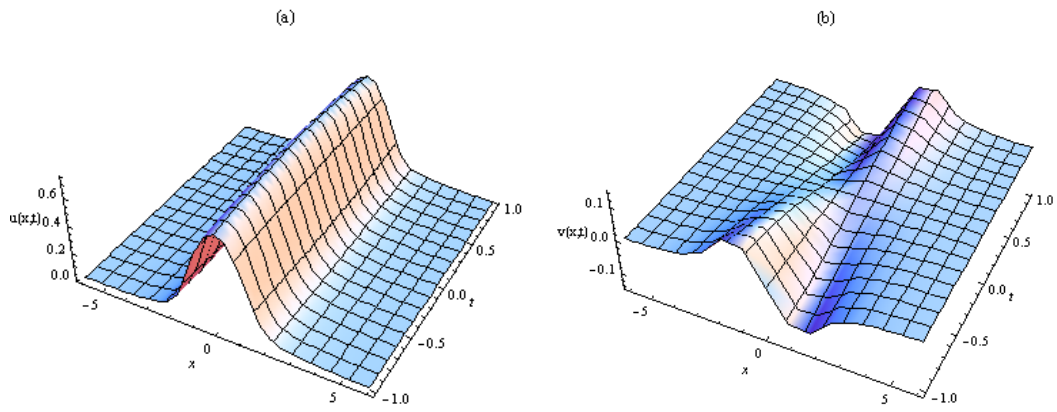


Figure 1: The surfaces show the 15th order approximate soliton solutions to (a) the height $u(x,t)$ and (b) the velocity $v(x,t)$ of water for $|x| \leq 6$ and $|t| \leq 1$.

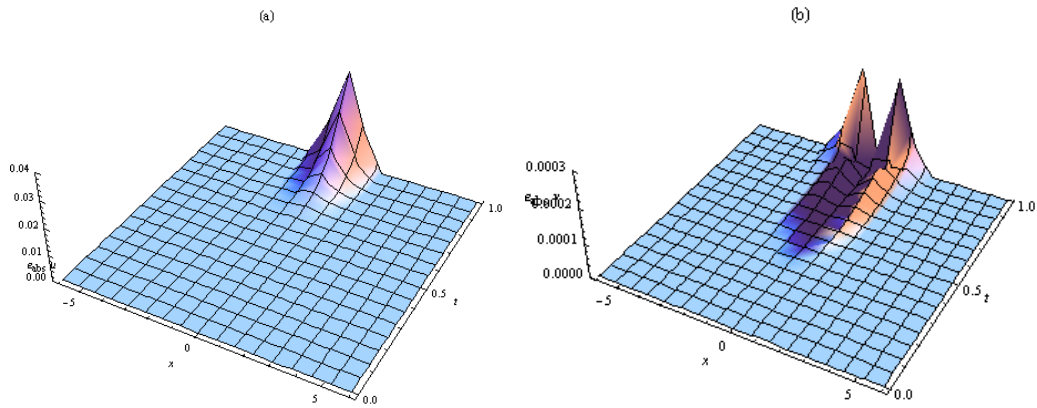


Figure 2: The absolute errors using 15-terms approximate solutions of (a) the height $u(x,t)$ and (b) the velocity $v(x,t)$ of water for $|x| \leq 6$ and $|t| \leq 1$.

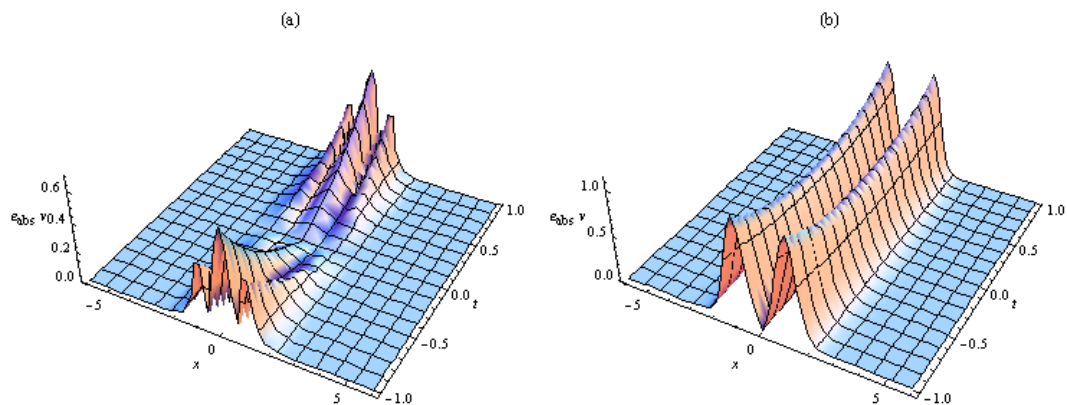


Figure 3: The absolute errors using the ADM of (a) the height $u(x,t)$ and (b) the velocity $v(x,t)$ of water for $|x| \leq 6$ and $|t| \leq 1$.

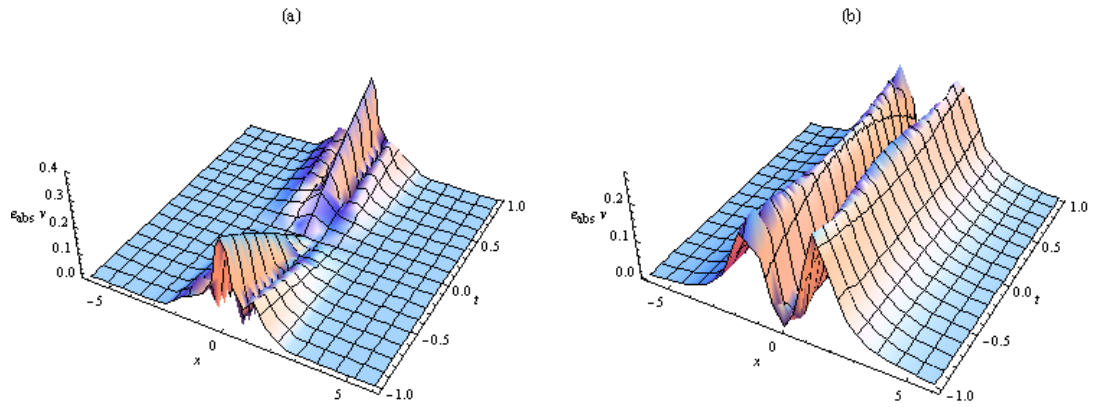


Figure 4: The absolute errors using the VIM of (a) the height $u(x,t)$ and (b) the velocity $v(x,t)$ of water for $|x| \leq 6$ and $|t| \leq 1$.

The errors can be made smaller as more and more terms of the series solution are included. To show the advantages and accuracy of the presented method for solving shallow water equations, Figures 3 and 4 represent the absolute errors obtained by solving Eq. (1) using the Adomian decomposition method (ADM) and the variational iteration method (VIM) respectively. In view, it is obvious that the RDTM is reliable and of high accuracy in obtaining an analytic solutions for this system.

The differential transform method [17] is applied to solve the system of shallow water equations (1), the method is divergent except for small values of x and t . In addition, it is more divergent while more components of the series solutions are computed.

5. Conclusions

The reduced differential transform method has been applied to find an approximate solution of the one-dimensional shallow water equations. The method is low cost and easy to handle which produces high accurate solutions in comparisons with the Adomian decomposition method, the variational iteration method and the differential transform method. The algorithm can be used to provide an analytic solution of physical applications modeled by nonlinear partial differential equations.

To overcome the demerit of complex calculations of differential transform method, The reduced differential transform method was presented. Also, it presents an efficient improvement in solving nonlinear partial differential equations since the amount of computations required is much less than that in other existing techniques. Its rapid convergence, gives exact solution with small number of iterations.

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