International Journal of Sciences: Basic and Applied Research (IJSBAR)

ISSN 2307-4531
(Print & Online)

http://gssrr.org/index.php?journal=JournalOfBasicAndApplied

Log-Beta Log-Logistic Regression Model

Mahmoud Riad Mahmoud[a], A. A. EL- Sheikh[b], Naglaa A. Morad[c], Moshera A. M. Ahmad[d*]

[a]*Dept. of Mathimatical Statistics, Institute of Statistical Studies and Research, Cairo University.*

[b,c,d]*Dept. of Applied Statistics and Econometrics, Institute of Statistical Studies and Research ,Cairo University.*

[a]*Email: dr.mahmoud.riad@hotmail.om*

[b]*Email: aham103@yahoo.om*

[c]*Email: naglaa_morad@yahoo.com*

[d]*Email: moshera_ahmad@pg.cu.edu.eg*

Abstract

In this article, the log beta log-logistic regression model based on the beta log-logistic distribution is which has a wider range of applications. The estimates of the parameters of the model for censored data are derived. Finally, the proposed model is applied to a real data set. Model checks based on martingale residuals and the *AIC* and *BIC* statistics are used to suggest appropriate models.

Keywords: beta log-logistic distribution; censored data; profile log-likelihood; survival function; lifetime data; maximum likelihood estimation; martingale residuals.

1. Introduction

The statisticians have interested in constructing flexible distributions to facility better modeling of lifetime data. So they made generalization of some lifetime models such as generalized log gamma, exponentiated Weibull, modified Weibull, and β-Birnbaum-Saunders distributions. Although many distributions are discussed in the literature, few regression models have been proposed.

---

* Corresponding author.

E-mail address: moshera_ahmad@pg.cu.edu.eg.

Regression models can be proposed in different forms in survival analysis; for example, the location-scale regression model which is frequently used in clinical trials. In this paper, we introduce a location-scale regression model, which will be referred to as log-beta log-logistic regression model based on a recently introduced continuous distribution, proposed by Lemonte [13] that extend the log- logistic distribution and some other distributions. Lemonte [13] named this distribution beta log-logistic distribution. The main motivation for the use of the log-beta log-logistic regression model is that it is much more flexible than the log-logistic regression model, i.e., the additional shape parameters (*a* and *b*) allow for a high degree of flexibility of the log-beta log-logistic regression model. So the new regression model can be helpful in many practical situations for modeling positive real data sets.

The article is organized as follows: In Section 2, we present the beta log-logistic distribution proposed by Lemonte [13]. In Section 3, we propose a log beta log-logistic regression model of location-scale form, estimate the model parameters by maximum likelihood and derive the observed information matrix. Residuals analysis is presented in Section 4. We show in Section 5 that the proposed model is more adequate to fit the myeloma patients data set, given by Krall and his colleagues [10], than log-beta Weibull (LBW) regression model proposed by Ortega and his colleagues [16], by checking the residual plots for both models and discriminating between the models using the *AIC* and *BIC* statistics. Finally, Section 6 offers some concluding remarks.

2. Beta log-logistic distribution

Lemonte [13], using the generator approach suggested by Eugene and his colleagues [7], defined a new statistical model which he called the beta log-logistic (BLLog) distribution. This new distribution generalizes the log-logistic (LLog) model. The basic idea of the generator approach is as follows. Starting with a distribution function $G(\cdot)$ and a random variable $Y \sim B(a,b)$, another random variable T is defined by $Y = G(T)$. The distribution of the random variable T is called the beta-G distribution. The distribution of T is given by

$$F(t) = pr[T \leq t] = pr[Y \leq G(t)] = I_{G(t)}(a,b) = \frac{1}{B(a,b)} \int_0^{G(t)} \varpi^{a-1} (1-\varpi)^{b-1} d\varpi. \quad (1)$$

This new distribution $F(t)$ adds new parameters $a > 0$ and $b > 0$ to those already in $G(T)$. Here, $B(p;\ q) = \Gamma(p)\,\Gamma(q)/\,\Gamma(p+q)$ is the beta function, where $\Gamma(.)$ is the gamma function. $B_y(p,q) = \int_0^y \varpi^{p-1}(1-\varpi)^{q-1} d\varpi$ is the incomplete beta function and $I_y(p;\ q) = B_y(p;\ q)/B(p;\ q)$ is the regularized(incomplete) beta function. The added parameters a and b may be used to modify the shape and the skewness of the distribution. The probability density function (pdf) corresponding to (1) is given by

$$f(t) = \frac{g(t)}{B(a,b)} G(t)^{a-1}[1-G(t)]^{b-1}, \quad (2)$$

where $g(t)$ is the probability density function corresponding to $G(t)$.

The hazard rate function associated with (1) is defined as

$$r(t) = \frac{g(t)G(t)^{a-1}[1-G(t)]^{b-1}}{B(a,b)[1-I_{G(t)}(a,b)]}.$$

The log logistic distribution is the probability distribution of a random variable whose logarithm has a logistic distribution, this distribution is also known as Fisk distribution which was first introduced by Champernowne [3] for graduating income distribution. It has a cdf of the form

$$G(t) = \frac{t^{\delta}}{\alpha^{\delta} + t^{\delta}}, \quad t > 0, \tag{3}$$

where $\alpha > 0$ and $\delta > 0$ are scale and shape parameter respectively. This distribution is used in survival analysis as a parametric model for events whose hazard rate increases initially and decreases later. The pdf corresponding to (3) is given by

$$g(t) = \frac{\delta\left(\frac{t}{\alpha}\right)^{\delta-1}}{\alpha\left[1+\left(\frac{t}{\alpha}\right)^{\delta}\right]^{2}}, \qquad t > 0. \tag{4}$$

Lemonte [13], inserting (3) and (4) in (2), obtained the beta log-logistic (BLLog) density function with positive parameters a, b, α and δ. The pdf of BLLog(a; b; α; δ) is given by

$$f(t) = \frac{(\frac{\delta}{\alpha})(\frac{t}{\alpha})^{a\delta-1}}{B(a,b)[1+(\frac{t}{\alpha})^{\delta}]^{a+b}}, \qquad t > 0. \tag{5}$$

The cdf corresponding to (5) is

$$F(t) = I_{\frac{t^{\delta}}{\alpha^{\delta}+t^{\delta}}}(a,b),$$

the survival function is

$$S(t) = 1 - I_{\frac{t^{\delta}}{\alpha^{\delta}+t^{\delta}}}(a,b),$$

and the associated hazard rate function takes the form

$$h(t) = \frac{\left(\frac{\delta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{a\delta-1}\left(1+(\frac{t}{\alpha})^{\delta}\right)^{-(a+b)}}{B(a,b)S(t)}, t > 0. \tag{6}$$

The new density (5) includes The LLog distribution as special cases. The LLog distribution arises when $a = b = 1$.

3. Log beta log-logistic regression model

Although lifetime of different individuals follows the same probability distribution, the parameters of the distribution may change depending on certain characteristic of the individual, variables representing these characteristic are referred to as covariates or explanatory variables, for example the mean value of the lifetime of an individual may depend on the blood pressure, sex, and weight. So, it would be interest to find the relationship between the lifetime and the explanatory variables. The most common approach to this type of relationship is a regression model.

3.1 Location –scale regression model

The class of location-scale models will be considered. The covariates vector is denoted by $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ip})^T$, which is related to responses $Y = \log(T)$ through a regression model.

Considering reparametrization of $f(t)$ in (5), $\delta = 1/\sigma$ and $\alpha = exp(\mu)$, it follows that the density function of Y can be written as

$$f(y; a, b, \sigma, \mu) = \frac{\sigma^{-1}}{B(a,b)}\left[exp\left(\frac{y-\mu}{\sigma}\right)\right]^a\left[1+exp\left(\frac{y-\mu}{\sigma}\right)\right]^{-(a+b)}, \quad (7)$$

where $a, b, \sigma > 0$, $-\infty < \mu < \infty$, and $-\infty < y < \infty$. The survival function takes the form

$$s(y) = 1 - I_{\frac{exp((y-\mu)/\sigma)}{[1+exp((y-\mu)/\sigma)]}}(a,b)\,. \quad (8)$$

The hazard rate function is, then, given by

$$h(y) = \frac{\frac{\sigma^{-1}}{B(a,b)}\left[exp\left(\frac{y-\mu}{\sigma}\right)\right]^a\left[1+exp\left(\frac{y-\mu}{\sigma}\right)\right]^{-(a+b)}}{1 - I_{\frac{exp((y-\mu)/\sigma)}{[1+exp((y-\mu)/\sigma)]}}(a,b)}, \quad (9)$$

Similar to Lawless[11], Ortega and his colleagues [15], Silva and his colleagues [19], Carrasco and his colleagues [2], Silva and his colleagues [20], Silva and his colleagues [18], Hashimotto and his colleagues [9], Gusmao and his colleagues [8], and Cordeiro and his colleagues [5] and others, we propose another way of expressing the dependence of y_i on , $\mathbf{x}_i$ as

$$y_i = \mathbf{x}^T_i\boldsymbol{\beta} + \sigma z_i\,, \qquad i = 1, \ldots, n \quad (10)$$

where y_i is the response variable , $\mathbf{x}^T_i = (x_{i1}, x_{i2}, \ldots, x_{ip})$ is the vector of explanatory variable , $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^T$, $\sigma > 0$, and z_i is a random error with density function (11)

$$f(z) = \frac{1}{B(a,b)}[exp(z)]^a[1+\exp(z)]^{-(a+b)}, \qquad -\infty < z < \infty. \quad (11)$$

The parameter $\mu_i = \mathbf{x}^T_i \boldsymbol{\beta}$ is the location of y_i. Where vector $\boldsymbol{\mu} = (\mu_1,..., \mu_n)^T$ and, $\boldsymbol{X} = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n)$ is a known model matrix.

3.2. *Estimation of the model parameters*

Let t_i denote the survival time of an individual or an item under observation, and suppose that these survival times are influenced by the regressor vector $\mathbf{x}_i$. However, not all the survival times are observed; some t_i's will be censored at time c_i. Let δ_i be an indicator variable denoting whether the i th observation was observed (δ_i =1) or censored (δ_i =0). Consider a sample $(y_1,\mathbf{x}_1),\ldots,(y_n,\mathbf{x}_n)$ of n independent observations, where $y_i = \delta_i \log(t_i)+(1-\delta_i) \log(c_i)$. We assume noninformative censoring such that the observed lifetimes and censoring times are independent. the log likelihood function for the vector of parameters $\boldsymbol{\theta}=(a,b,\delta,\boldsymbol{\beta}^T)^T$ from model (10) takes the form

$$l(\theta) = \sum_{i=1}^{n} \delta_i \, log[f(y_i)] + \sum_{i=1}^{n} (1-\delta_i) log[S(y_i)],$$

where $f(y_i)$ is the density function (7) and $S(y_i)$ is the survival function (8) of Y_i. The log-likelihood function for $\boldsymbol{\theta}$ reduces to

$$l(\theta) = -rlog(\sigma) - rlogB(a,b) + a \sum_{i=1}^{n} \delta_i (z_i) - (a+b) \sum_{i=1}^{n} \delta_i (log[1+exp(z_i)]) + \sum_{i=1}^{n} (1-\delta_i) log\left\{1 - I_{\frac{exp(z_i)}{[1+exp(z_i)]}}(a,b)\right\}, \quad (12)$$

where r is the number of uncensored observations (failures) and $z_i = (y_i - \mathbf{x}^T_i \boldsymbol{\beta})/\sigma$. The Maximum likelihood estimates $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ can be obtained by maximizing the log-likelihood function (12).

Let $I(\boldsymbol{\theta}) = E\left[\ddot{L}(\boldsymbol{\theta})\right]$ is the observed information matrix and the asymptotic covariance matrix $\boldsymbol{\Gamma}^1(\boldsymbol{\theta})$ of $\hat{\theta}$ can be approximated by the inverse of the (p+3)(p+3) observed information matrix $\ddot{L}(\widehat{\boldsymbol{\theta}}) = -\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\theta\partial\theta^T}|_{\theta=\hat{\theta}}$.

$$-\ddot{\boldsymbol{L}}(\boldsymbol{\theta}) = \begin{pmatrix} L_{aa} & L_{ab} & L_{a\sigma} & L_{a\beta_j} \\ . & L_{bb} & L_{b\sigma} & L_{b\beta_j} \\ . & . & L_{\sigma\sigma} & L_{\sigma\beta_j} \\ . & . & . & L_{\beta_j\beta_s} \end{pmatrix}$$

where $L_{a\beta_j} = [L_{a\beta_1} \quad ... \quad L_{a\beta_p}]$, $L_{b\beta_j} = [L_{b\beta_1} \quad ... \quad L_{b\beta_p}]$, $L_{\sigma\beta_j} = [L_{\sigma\beta_1} \quad ... \quad L_{\sigma\beta_p}]$, and

$$L_{\beta_j\beta_s} = \begin{bmatrix} L_{\beta_1\beta_1} & ... & L_{\beta_1\beta_p} \\ \vdots & \ddots & \vdots \\ L_{\beta_p\beta_1} & ... & L_{\beta_p\beta_p} \end{bmatrix}$$

4. Residual analysis

After the model is fitted, we need a tool to check the assumptions and assess the adequacy of the fitted model. The examination of residuals is an important way to check assumptions in the fitted model. In survival analysis with right censored data, use of martingale residuals which were proposed by Barlow and Prentice [1] is one way to assessing leverage and goodness of fit. In parametric lifetime models, we can define the martingale residual as (see for example, Therneau and his colleagues [17], Commenges and Rondeau [4], and Elgmati [6]) the difference between the counting process and the integrated intensity function (which is also known hazard rate function given in ,9),

$$r_{M_i} = \delta_i - \left(\int_0^y h(u)du\right), \qquad i=1,\ldots,n$$

where δ_i take the value 0 or 1 if the ith observation is censored or uncensored respectively.

As we known $\int_0^y h(u)du = -\log[S(y)]$ therefore it can be reduces to the simple form

$$r_{M_i} = \delta_i + \log\left[S\left(y_i, \hat{\theta}\right)\right],$$

(see for example, Ortega and his colleagues [14], Silva and his colleagues [20], and Hashimoto and his colleagues [9]).

The martingale residuals are skew, have maximum value +1 and minimum value $-\infty$. The martingale residual for the log beta log-logistic model takes the form

$$r_{M_i} = \begin{cases} 1 + log\left\{1 - I_{\frac{exp(\frac{y_i - \boldsymbol{X}_i^T\hat{\boldsymbol{\beta}}}{\hat{\sigma}})}{[1+exp(\frac{y_i - \boldsymbol{X}_i^T\hat{\boldsymbol{\beta}}}{\hat{\sigma}})]}}(\hat{a},\hat{b})\right\} if\ \delta_i = 1, \\ \left\{1 - I_{\frac{exp(\frac{y_i - \boldsymbol{X}_i^T\hat{\boldsymbol{\beta}}}{\hat{\sigma}})}{[1+exp(\frac{y_i - \boldsymbol{X}_i^T\hat{\boldsymbol{\beta}}}{\hat{\sigma}})]}}(\hat{a},\hat{b})\right\} if\ \delta_i = 0. \end{cases}$$

5. Application

In order to demonstrate the proposed methodology, we use the myeloma patients data set given by Krall and his colleagues [10] and a subset of which is reported in Lawless [12]. The aim of our study is to relate the logarithm of the survival time ($y=log\ t$) for multiple myeloma to a number of prognostic variables for censored data. The data reports the survival times (t), in months, for 65 patients with multiple myeloma who were treated with a certain drug. Out of these patients only 17 survived to the end of the study, while 48 died during the study. The

data includes several possible explanatory variables but only five of them [as in Lawless] are used in the following analysis. These variables are: logarithm of a blood urea nitrogen measurement at diagnosis (x_1), hemoglobin measurement at diagnosis (x_2), age at diagnosis (x_3), sex (x_4) [0 for male and 1 for female], and serum calcium measurement at diagnosis (x_5).

Except for the sex (x_4), the data was centered and the model

$$y_i = \beta_0 + \beta_1(x_{i1} - \bar{x}_1) + \beta_2(x_{i2} - \bar{x}_2) + \beta_3(x_{i3} - \bar{x}_3) + \beta_4 x_{i4} + \beta_5(x_{i5} - \bar{x}_5) + \sigma z_i,$$

was employed, where the variable $y_i = log(\ t_i\ )$ is assumed to follow the log BLLog distribution given in (7), in which the random errors z_i have density function (11).

When we tried to maximize the likelihood function (12) numerically we found that in many cases the numerical procedure failed to converge and negative values of σ were produced. As an alternative we use the profile log-likelihood approach as describe by Rao [21] as follows.

Suppose a is known, then we rewrite the log-likelihoods $L(\theta)=L_a(b, \sigma, \beta)$ (to show that a is fixed but b, σ, and β vary). The profile likelihood of a can be defined as

$$Pl(a) = \left(\tilde{b}(a), \tilde{\sigma}(a), \tilde{\beta}(a)\right) \equiv \arg max_{b,\sigma,\beta}\ L_a(\ b, \sigma, \beta).$$

It means that we maximize $L_a(\ b,\sigma, \beta)$ with respect to b, σ, and β to estimate b, σ, and β. To a large extent the profile likelihood could be used as a full likelihood and it should be maximized with respect to a. In general, we may not have an analytic formula for $Pl(a)$; only a numeric value corresponding to a numeric specification of a. Using a large set of values of a and their corresponding values for $Pl(a)$ we can construct the graph $(a, Pl(a))$. From this graph we can obtain an approximation of the value of a that maximizes $Pl(a)$. In other words we evaluate

$$\hat{a} = \arg max_a\ L_a = \left(\hat{b}, \hat{\sigma}, \hat{\beta}\right) = \arg max_a \mathrm{L}\left(\mathrm{a}, \tilde{b}(a), \tilde{\sigma}(a), \tilde{\beta}(a)\right).$$

Thinking about it we can see that $\hat{a}$ and $\hat{b}, \hat{\sigma}, \text{and } \hat{\beta}$ are the maximum likelihood estimators $\hat{\theta} = \left(\hat{a}, \hat{b}, \hat{\sigma}, \hat{\beta}\right) = \arg max_\theta\ L(\theta)$.

To illustrate that, we do the following steps:

1. For our application, get the initial values for b,σ, and β from the fit of the log beta weibull regression model (see, Ortega and his colleagues [12])

2. For every value of a in an appropriate set that is thought to contain the maximum likelihood $\hat{a}$ of a, calculate the MLEs $\tilde{b}(a), \tilde{\sigma}(a), \text{and } \tilde{\beta}(a)$ conditioned on a, and then the maximized log-likelihood function $L_{max}(a)$ is determined.

3. Maximize the log-likelihood $L_{max}(a)$, to obtain $\hat{a}$. There for the MLEs of b, σ, and β are given by $\hat{b} = \tilde{b}(a), \hat{\sigma} = \tilde{\sigma}(a),$ and $\hat{\beta} = \tilde{\beta}(a),$ respectively. Figure 1 shows that the profile log-likelihood $l(\hat{b}(a), \hat{\sigma}(a),$ and $\hat{\beta}(a))$ reaches its maximum value at a = 170. Hence, this value is taken as the MLE of a.

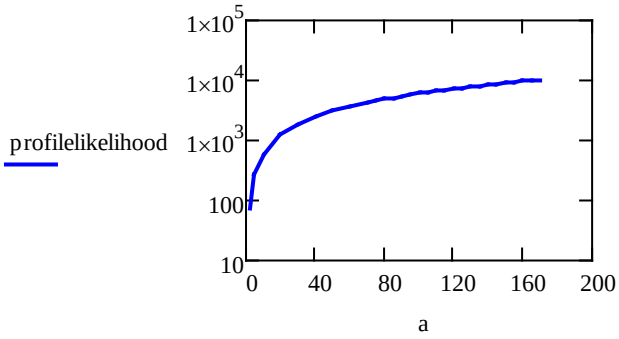

Figure 1: Maximized profile log likelihood for the log BLLog regression model to the myeloma data.

Now, for the myeloma patients data, we would like to compare between log-beta log logistic and log-beta weibull regression models. To choose between competing models we use Akaike's information criterion (*AIC*) and Bayesian information criterion (*BIC*) statistics, there are defined as

AIC=-2.*log*(*likelihood*)+2(*p*+2+*k*) and *BIC*=-2 *log*(*likelihood*)+(*p*+*k*)*log*(*n*)

where p is the number of estimated parameters and k = 2 is an arbitrary constant for both models. Lower values of the *AIC* and *BIC* indicate the preferred model.

We fitted the log beta log logistic and the log-beta weibull regression models to myeloma patient's data. Table 1 gives the estimates and their standard errors of the parameters for both regression models. The values of the statistics *AIC* and *BIC* are then used to select the better model. The statistic *AIC* yields the value -2.252×10^4 for the log-beta log logistic regression model and 218.194 for the log-beta weibull regression model , whereas the statistic *BIC* yields -2.252×10^4 for the log-beta log logistic regression model and 212.136 for the log-beta weibull regression model. The values of these statistics indicate that the log-beta log logistic regression model is more adequate to explain the data set than the log-beta weibull regression model.

The current estimates of the regression parameters for the log **BLLog** regression model and the **LBW** regression model and their standard errors are represented in Table 1. We can note that the log **BLLog** regression model has standard errors smaller than the **LBW** regression model.

In order to detect possible outlying observations as well as departures from assumptions of the log **BLLog** regression model and the **LBW** regression model we present. In figures 2 and 3, the graphs of martingale residuals against y, log time. By analyzing these graphs, asymmetry is observed, since, we know that the range of the martingale residuals is between (-∞,1), and we show in our plots it is between (0,1) and as we show

there is no outliers in the martingale plots and and the both figures seem to fit the data very well.

Table 1: Estimates of the parameters ,standard errors , P-values , and confidence intervals in (.) for the Log BLLog and LBW models fitted to the myeloma data.

parameter	BLLog			LBW		
	Estimate	SE	P-values	Estimate	SE	P-values
b	0.093 (0.053,0.133)	0.02		0.181 (0.057, 0.305)	0.063	
σ	1 (0.864, 1.136)	0.069		2.012 (1.462, 2.562)	0.281	
β_0	0.09 (-0.262, 0.442)	0.18	0.5	0.288 (-2.135, 2.710)	1.236	0.816
β_1	-0.396 (-0.759,-0.033)	0.185	0.02	-1.564 (-2.280, -0.849)	0.365	<0.001
β_2	0.085 (0.036, 0.134)	0.025	0.001	0.161 (0.071, 0.252)	0.046	< 0.001
β_3	0.001253 (-0.009375,0.012)	0.005	0.5	0.005 (-0.018, 0.029)	0.012	0.654
β_4	0.121 (-0.145, 0.387)	0.136	0.4	0.280 (-0.246, 0.805)	0.268	0.297
β_5	-0.039 (-0.094, 0.016)	0.028	0.1	-0.154 (-0.272, -0.036)	0.060	0.011

The log BLLog model involves two extra parameters which gives it more flexibility to fit the data. The explanatory variables x_1 and x_2 are marginally significant for the log BLLog model at the significance level of 5%. We note from the fitted log BLLog regression model that the age at diagnostic and the sex do not seem to be significant. Therefore, it is to be expected that an individual with low blood urea nitrogen and the serum calcium measurements at diagnosis would survive longer, while an individual with high hemoglobin measurement at diagnosis would survive longer.

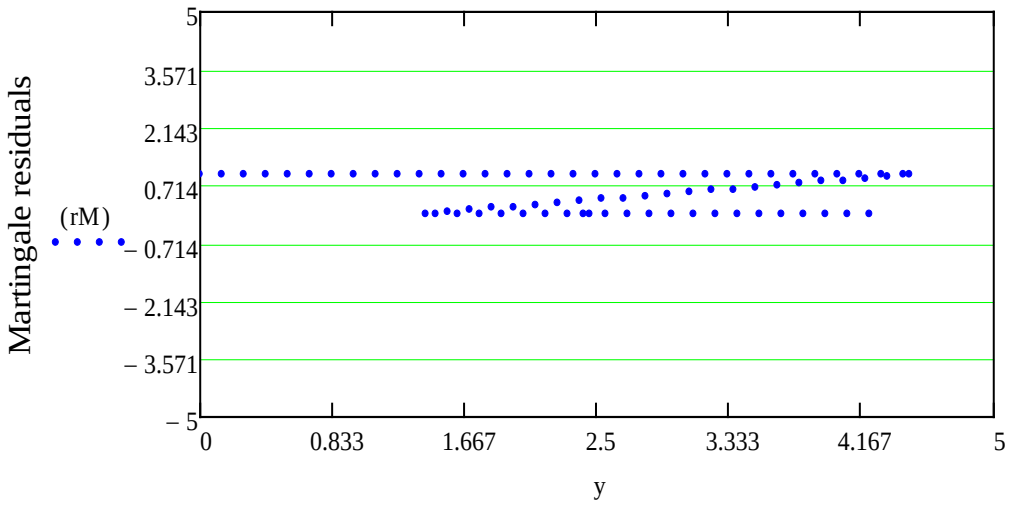

Figure 2: Plot of the Martingale residuals against y for the log BLLog model

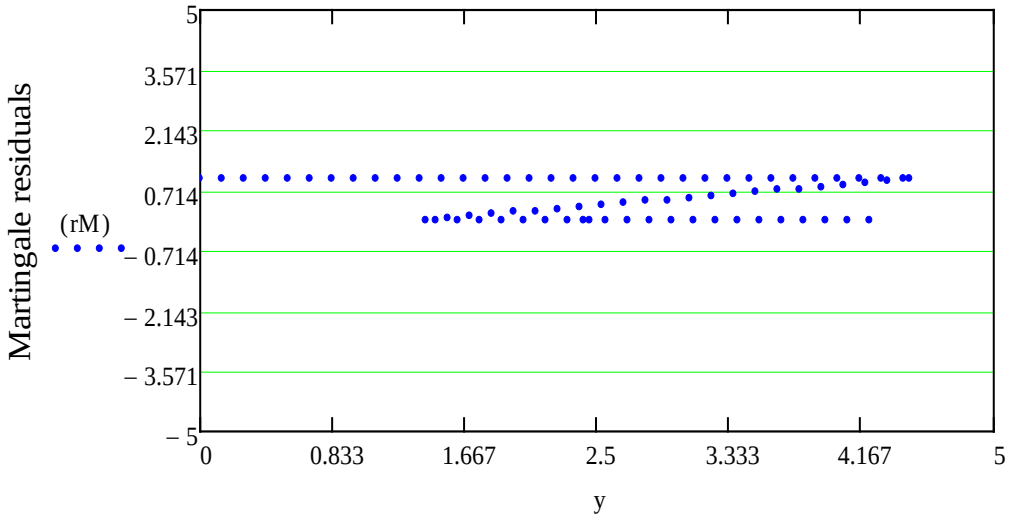

Figure 3: Plot of the Martingale residuals against y for the LBW model

The final model will be as follow

$$y_i = \beta_0 + \beta_1(x_{i1} - \bar{x}_1) + \beta_2(x_{i2} - \bar{x}_2) + \sigma z_i,$$

The parameter estimates in the final model are given in table 2. The estimates can be interpreted as following; the median survival time should increase approximately 8% ($e^{0.085}$ × 100%) as the hemoglobin measurement increases one unit.

Table 2: MLEs of the parameters from the log BLLog regression model on the myeloma data set – final model

parameter	Estimate	S.E.	p-value	C.I. 95%
b	0.093	0.02	-	(-0.184, 0.37)
σ	1	0.059	-	(0.524, 1.476)
β_0	0.091	0.149	0.5	(-0.666, 0.848)
β_1	-0.396	0.188	0.02	(-1.246, 0.454)
β_2	0.085	0.023	0.001	(-0.212, 0.382)

6. Concluding remarks

According to the previous results it can be seen that the log BLLog regression model is more representable for modeling censored and uncensored lifetime data. The proposed model serves as an important extension to several existing regression models. Hence, the proposed regression model can be considered as an alternative model for lifetime data analysis and be more flexible than the LLog model (arises as the basic exemplar when $a = b = 1$). Maximum likelihood is described for estimating the model parameters, and the usefulness of the model is also demonstrated through the analysis of a real data set.

References

[1] Barlow, W. E. and Prentice, R. L. Residuals for relative risk regression. *Biometrika*, vol. 75, 1, 65-74, 1988.

[2] Carrasco,J.M.F., Ortega, E.M.M.,and Paula,G.A. Log-modified Weibull regression models with censored data: Sensitivity and residual analysis. *Computational Statistics and Data Analysis*, vol.52, 4021–4039, 2008.

[3] Champernowne, D.G. The graduation of income distributions. *Econometrica*, vol.20, 4, 591-615, 1952.

[4] Commenges, D. and Rondean, V. Standardized martingale residuals applied to grouped left truncated observations of dementia cases. *Lifetime data analysis*, vol. 6, pp. 229-235, 2000.

[5] Cordeiro, G.M., Hashimoto,E.M., and Ortega, E.M.M. The McDonald Weibull model. Statistics, *iFirst*, pp.1–23, 2012.

[6] Elgmati, E. A. A martingale residuals diagnostic for logistic regression model. *World academy of science, engineering and technology*, vol.6, pp.465-469, 2012.

[7] Eugene, N., Lee, C.,Famoye, F. Beta-normal distribution and its applications. *Communications in Statistics – Theory and Methods*, vol.31, pp.497–512, 2002.

[8] Gusmão,FRS, Ortega, E.M.M.,and Cordeiro, G.M. The generalized inverseWeibull distribution. *Stat Papers*. vol. 52, pp.591–619, 2011.

[9] Hashimoto,E.M., Ortega, E.M.M., Cancho,V,G.,and Cordeiro, G.M. The log-exponentiated Weibull regression model for interval-censored data. *Computational Statistics and Data Analysis, vol*. 54, pp.1017-1035, 2010.

[10] Krall,J., Uthoff,V., and Harley,J. A step-up procedure for selecting variables associated with survival, *Biometrics,* vol. 31, pp. 49-57, 1975.

[11] Lawless, J.F. *Statistical Models and Methods for Lifetime Data*. John Wiley & Sons, 1982.

[12] Lawless, J.F. *Statistical Models and Methods for Lifetime Data*. 2nd ed, John Wiley, New York, 2003, pp.334.

[13] Lemonte,A.J. The beta log-logistic distribution. *Brazilian journal of probability and statistics*, vol.28, 3, pp.313-332, 2014.

[14] Ortega, E.M.M., Bolfarine,H., and Paula,G.A. Influence diagnostics in generalized log-gamma regression models. *Computational Statistics and Data Analysis*. vol. 42, pp. 165–186, 2003.

[15] Ortega, E.M.M., Cancho,V,G., and Bolfarine,H. Influence diagnostics in exponentiated-Weibull regression models with censored data. *SORT*, vol.30, 2, pp.171-192, 2006.

[16] Ortega, E.M.M., Cordeiro, G.M., and Hashimoto, E.M. A Log-Linear Regression Model for the Beta-Weibull Distribution. *Communications in Statistics—Simulation and Computation*, vol.40, pp.1206–1235, 2011.

[17] Therneau, T. M., Grambsch, P. M. and Fleming, T. R. Martingale-based residuals for survival models. *Biometrika*, vol.77, 1, pp.147-160, 1990.

[18] Silva, G.O., Ortega, E.M.M.,and Cancho,V,G. Log-Weibull extended regression model: Estimation, sensitivity and residual analysis. *Statistical Methodology*, vol.7, pp.614-631, 2010.

[19] Silva, G.O., Ortega, E.M.M., Cancho,V,G.,and Barreto,M.L. Log-Burr XII regression models with censored data. *Computational Statistics and Data Analysis*, vol.52, pp.3820–3842, 2008.

[20] Silva, G.O., Ortega, E.M.M., Cordeiro, G.M. A log-extended Weibull regression model. *Computational Statistics and Data Analysis*. vol.53, pp.4482-4489, 2009.

[21] Rao, S. S.(2012). *Lecture notes on Advanced statistical inference.* Available: http://www.stat.tamu.edu/~suhasini/teaching613/teaching613_spring2012.html, Last visited on 7-5-2015, 5.50pm.

Appendix: Matrix of second derivatives $-\ddot{L}(\boldsymbol{\theta})$

The necessary formulas to obtain the second-order partial derivatives of the log-likelihood function are given after some algebraic manipulations, the following formulas are obtained

$$L_{aa} = -r\,\psi_{(a)}^{(1)} + r\psi_{(a+b)}^{(1)} - \sum_{\mathcal{C}} \frac{\left(\frac{1}{B(a,b)}\left(G(z_i)\right)^{a}\Gamma_{(a)}^{2}o_i - I_{(G(z_i))}(a,b)D_i\right)^{2}}{\left[1 - I_{(G(z_i))}(a,b)\right]^{2}} + \frac{\zeta 1}{\left[1 - I_{(G(z_i))}(a,b)\right]},$$

$$L_{bb} = -r\psi_{(b)}^{(1)} + r\psi_{(a+b)}^{(1)} + \sum_{\mathcal{C}} \frac{-\left[\left(1 - G(z_i)\right)^{b}\Gamma_{(b)}^{2}v_i + B_{(1-G(z_i))}(a,b)d_i\right]^{2}}{B^{2}(a,b)\,(1 - I_{(G(z_i))}(a,b))^{2}} + \frac{\left(\left(1 - G(z_i)\right)^{b}\Gamma_{(b)}^{2}v_i + B_{(1-G(z_i))}(a,b)d_i\right)\left(\psi(b) - \psi(a+b)\right)}{\left[B(a,b)(1 - I_{(G(z_i))}(a,b))\right]} - \frac{\zeta 2}{\left[B(a,b)(1 - I_{(G(z_i))}(a,b))\right]},$$

$$L_{ab} = r\psi_{(a+b)}^{(1)} + \frac{\sum_{\mathcal{C}}\left[\left\{\left(1-G(z_i)\right)^{b}\Gamma_{(b)}^{2}v_i + B_{(1-G(z_i))}(a,b)d_i\right\}\left\{\frac{1}{B(a,b)}\left(G(z_i)\right)^{a}\Gamma_{(a)}^{2}[o_i] - B_{(1-G(z_i))}(a,b)D_i\right\}\right]}{\sum_{\mathcal{C}}\left\{B(a,b)\left(1 - B_{(G(z_i))}(a,b)\right)^{2}\right\} + \frac{\zeta 3}{\sum_{\mathcal{C}}\left[1 - B_{(G(z_i))}(a,b)\right]}},$$

$$\begin{aligned} L_{a\beta} = {} & \sum_{\mathcal{F}} \frac{x_i e^{z_i}}{\sigma[1 + e^{z_i}]} - \sum_{\mathcal{F}} \frac{x_i}{\sigma} - \sum_{\mathcal{C}} \frac{\left(G(z_i)\right)^{(a-1)}\left(1 - G(z_i)\right)^{(b-1)}(\xi 1_i)log\left(G(z_i)\right)}{[1 - I_{(G(z_i))}(a,b)]} \\ & + \sum_{\mathcal{C}} \frac{\left(G(z_i)\right)^{(a-1)}\left(1 - G(z_i)\right)^{(b-1)}(\xi 1_i)(\psi(a) - \psi(a+b))}{B(a,b)[1 - I_{(G(z_i))}(a,b)]} \\ & + \sum_{\mathcal{C}} \frac{\left(G(z_i)\right)^{(a-1)}\left(1 - G(z_i)\right)^{(b-1)}(\xi 1_i)\left[\frac{1}{B(a,b)}\left(G(z_i)\right)^{a}\Gamma_{(a)}^{2}[o_i] - I_{(G(z_i))}(a,b)D_i\right]}{B(a,b)[1 - I_{(G(z_i))}(a,b)]^{2}}, \end{aligned}$$

$$\begin{aligned} L_{b\sigma} = {} & \sum_{\mathcal{F}} \frac{z_i e^{z_i}}{\sigma[1 + e^{z_i}]} \\ & - \sum_{\mathcal{C}} \frac{\left(G(z_i)\right)^{(a-1)}\left(1 - G(z_i)\right)^{(b-1)}(\xi 2_i)log\left(1 - G(z_i)\right)}{B(a,b)[1 - I_{(G(z_i))}(a,b)]} \\ & - \sum_{\mathcal{C}} \frac{\left(G(z_i)\right)^{(a-1)}\left(1 - G(z_i)\right)^{(b-1)}(\xi 2_i)\left[\left(1 - G(z_i)\right)^{(b-1)}\Gamma_{\mathrm{b}}^{2}v_i + B_{(1-G(z_i))}(a,b)d_i\right]}{B^{2}(a,b)\left[1 - B_{(G(z_i))}(a,b)\right]^{2}} \\ & + \sum_{\mathcal{C}} \frac{\left(G(z_i)\right)^{(a-1)}\left(1 - G(z_i)\right)^{(b-1)}(\xi 2_i)(\psi(b) - \psi(a+b))}{B(a,b)[1 - I_{(G(z_i))}(a,b)]}, \end{aligned}$$

$$L_{\sigma\beta}$$

$$= \frac{a}{\sigma^2}\sum_{\mathcal{F}} x_i - \frac{(a+b)}{\sigma^2}\sum_{\mathcal{F}} \frac{x_i e^{z_i}}{[1+e^{z_i}]^2} + \frac{(a+b)}{\sigma^2}\sum_{\mathcal{F}} \frac{x_i z_i e^{z_i}}{[1+e^{z_i}]} + \frac{(a+b)}{\sigma^2}\sum_{\mathcal{F}} \frac{x_i z_i e^{2(z_i)}}{[1+e^{z_i}]}$$

$$- \sum_{\mathcal{C}} \frac{\big(G(z_i)\big)^{2(a-1)}\big(1-G(z_i)\big)^{2(b-1)}(\xi 2_i)(\xi 1_i)}{B^2(a,b)[1-I_{(G(z_i))}(a,b)]}$$

$$- \sum_{\mathcal{C}} \frac{\left[\big(G(z_i)\big)^{(a-1)}\big(1-G(z_i)\big)^{(b-1)}\left[\frac{2x_i z_i e^{3(z_i)}}{\sigma^2[1+e^{z_i}]^3} - \frac{3x_i z_i e^{2(z_i)}}{\sigma^2[1+e^{z_i}]^2} + \frac{x_i z_i e^{(z_i)}}{\sigma^2[1+e^{z_i}]} - \frac{x_i e^{2(z_i)}}{\sigma^2[1+e^{z_i}]^2} + \frac{x_i e^{(z_i)}}{\sigma^2[1+e^{z_i}]}\right]\right]}{B(a,b)[1-I_{(G(z_i))}(a,b)]}$$

$$- \sum_{\mathcal{C}} \frac{(a-1)\big(G(z_i)\big)^{(a-2)}\big(1-G(z_i)\big)^{(b-1)}(\xi 2_i)(\xi 1_i)}{B(a,b)[1-I_{(G(z_i))}(a,b)]}$$

$$- \sum_{\mathcal{C}} \frac{(b-1)\big(G(z_i)\big)^{(a-1)}\big(1-G(z_i)\big)^{(b-2)}(-\xi 2_i)(\xi 1_i)}{B(a,b)[1-I_{(G(z_i))}(a,b)]},$$

$$L_{\sigma\sigma}$$

$$= \frac{r}{\sigma^2} + 2a\sum_{\mathcal{F}} \frac{z_i}{\sigma^2} - \frac{(a+b)}{\sigma^2}\sum_{\mathcal{F}} \frac{z_i^2 e^{z_i}}{[1+e^{z_i}]} + \frac{(a+b)}{\sigma^2}\sum_{\mathcal{F}} \frac{z_i^2 e^{2z_i}}{[1+e^{z_i}]^2} - \frac{2(a+b)}{\sigma^2}\sum_{\mathcal{F}} \frac{z_i e^{z_i}}{[1+e^{z_i}]}$$

$$- \sum_{\mathcal{C}} \frac{\big(G(z_i)\big)^{2(a-1)}\big(1-G(z_i)\big)^{2(b-1)}[\xi 2_i]^2}{B^2(a,b)[1-I_{(G(z_i))}(a,b)]^2}$$

$$- \sum_{\mathcal{C}} \frac{\big(G(z_i)\big)^{(a-1)}\big(1-G(z_i)\big)^{(b-1)}\left[\frac{2z_i^2 e^{3z_i}}{\sigma^2[1+e^{z_i}]^3} - \frac{3z_i^2 e^{2z_i}}{\sigma^2[1+e^{z_i}]^2} + \frac{z_i^2 e^{z_i}}{\sigma^2[1+e^{z_i}]} - \frac{2z_i e^{2z_i}}{\sigma^2[1+e^{z_i}]^2} + \frac{2z_i e^{z_i}}{\sigma^2[1+e^{z_i}]}\right]}{B(a,b)[1-I_{(G(z_i))}(a,b)]^2}$$

$$- \sum_{\mathcal{C}} \frac{(a-1)\big(G(z_i)\big)^{(a-2)}\big(1-G(z_i)\big)^{(b-1)}[\xi 2_i]^2}{B(a,b)[1-I_{(G(z_i))}(a,b)]^2} - \sum_{\mathcal{C}} \frac{(b-1)\big(G(z_i)\big)^{(a-1)}\big(1-G(z_i)\big)^{(b-2)}(\xi 2_i)(-\xi 2_i)}{B(a,b)[1-I_{(G(z_i))}(a,b)]^2},$$

$$L_{\beta\beta} = \frac{(a+b)}{\sigma^2}\sum_{\mathcal{F}} \frac{x_i e^{2z_i}}{[1+e^{z_i}]^2} - \frac{(a+b)}{\sigma^2}\sum_{\mathcal{F}} \frac{x_i^2 e^{z_i}}{[1+e^{z_i}]} - \sum_{\mathcal{C}} \frac{\big(G(z_i)\big)^{2(a-1)}\big(1-G(z_i)\big)^{2(b-1)}[\xi 1_i]^2}{B^2(a,b)[1-I_{(G(z_i))}(a,b)]^2}$$

$$- \sum_{\mathcal{C}} \frac{\big(G(z_i)\big)^{(a-1)}\big(1-G(z_i)\big)^{(b-1)}\left[\frac{2x_i^2 e^{3z_i}}{\sigma^2[1+e^{z_i}]^3} - \frac{3x_i^2 e^{2z_i}}{\sigma^2[1+e^{z_i}]^2} + \frac{x_i^2 e^{z_i}}{\sigma^2[1+e^{z_i}]}\right]}{B(a,b)[1-I_{(G(z_i))}(a,b)]}$$

$$- \sum_{\mathcal{C}} \frac{(a-1)\big(G(z_i)\big)^{(a-2)}\big(1-G(z_i)\big)^{(b-1)}[\xi 1_i]^2}{B(a,b)[1-I_{(G(z_i))}(a,b)]}$$

$$- \sum_{\mathcal{C}} \frac{(b-1)\big(G(z_i)\big)^{(a-1)}\big(1-G(z_i)\big)^{(b-2)}(\xi 1_i)(-\xi 1_i)}{B(a,b)[1-I_{(G(z_i))}(a,b)]},$$

$$L_{b\beta} = \sum_{\mathcal{F}} \frac{x_i e^{z_i}}{\sigma[1+e^{z_i}]} + \frac{\zeta 4}{\sum_{\mathcal{C}} B(a,b)[1-I_{(G(z_i))}(a,b)] + \frac{\sum_{\mathcal{C}}\left[(G(z_i))^{a-1}(1-G(z_i))^{(b-1)}(\xi 1_i)\left((1-G(z_i))^{b}\Gamma_{\mathrm{b}}^2 v_i + B_{(1-G(z_i))}(a,b)d_i\right)\right]}{\sum_{\mathcal{C}} B^2(a,b)\left[1-I_{(G(z_i))}(a,b)\right]^2}},$$

and

$$L_{a\sigma} = -\sum_{\mathcal{F}} \frac{z_i}{\sigma[1+e^{z_i}]} + \frac{\zeta 5}{\sum_{\mathcal{C}}\left[1-I_{(G(z_i))}(a,b)\right]} + \frac{\sum_{\mathcal{C}}\left[(G(z_i))^{a-1}(1-G(z_i))^{(b-1)}(\xi 2_i)\left[\frac{1}{B(a,b)}(G(z_i))^{a}\Gamma_{(a)}^2[o_i]\left[I_{(G(z_i))}(a,b)\right]D_i\right]\right]}{\sum_{\mathcal{C}} B(a,b)\left[1-I_{(G(z_i))}(a,b)\right]^2},$$

where

F,C are the sets of individuals for which y_i is the loglifetime and log-censoring, respectively.

$$z_i = \tfrac{y_i - x_i\beta}{\sigma},\ G(z_i) = \left(\tfrac{e^{z_i}}{1+e^{z_i}}\right),\ B(a,b) = \tfrac{\Gamma a \Gamma b}{\Gamma(a+b)} = \int_0^1 \varpi^a(1-\omega)^{b-1}d\varpi,$$

$$B_{(1-G(z_i))}(a,b) = \int_0^{(1-G(z_i))} \varpi^a(1-\omega)^{b-1}d\varpi,\ I_{(G(z_i))}(a,b) = \frac{1}{B(a,b)}\int_{(G(z_i))}^1 \varpi^a(1-\omega)^{b-1}d\varpi,$$

$$D_i = \left(log\left(G(z_i)\right) - \psi(a) + \psi(a+b)\right), d_i = \left(-\log\left(1-G(z_i)\right) + \psi(b) - \psi(a+b)\right),$$

$$o_i = \ {}_3F_2\left(a,a,1-b;1+a,1+a;G(z_i)\right), [o_i]_a = \partial_a\left[\ {}_3F_2\left(a,a,(1-b);(1+a),(1+a);G(z_i)\right)\right],$$

$${}_3F_2\left(a,a,1-b;1+a,1+a;G(z_i)\right) = \sum_{n=0}^{\infty} \frac{(a)_n(a)_n(1-b)_n}{(1+a)_n(1+a)_n}\frac{\left(G(z_i)\right)^n}{n!},$$

$$v_i = \ {}_3F_2\left(b,b,(a-1);(b+1),(b+1);\left(1-G(z_i)\right)\right),$$

$$[v_i]_b = \partial_b\left[\ {}_3F_2\left(b,b,(a-1);(b+1),(b+1);\left(1-G(z_i)\right)\right)\right],$$

$$q_i = \ {}_2F_1\left(a,(1-b);(1+a);G(z_i)\right),\ u_i = \ {}_2F_1\left(b,(1-a);(b+1);\left(1-G(z_i)\right)\right),$$

$$\xi 1_i = \left(\tfrac{x_i e^{2z_i}}{\sigma[1+e^{z_i}]^2} - \tfrac{x_i e^{z_i}}{\sigma[1+e^{z_i}]}\right),\ \xi 2_i = \left(\tfrac{z_i e^{2z_i}}{\sigma[1+e^{z_i}]^2} - \tfrac{z_i e^{z_i}}{\sigma[1+e^{z_i}]}\right),$$

$$\zeta 1 = \Bigg(\frac{1}{B(a,b)}\big(G(z_i)\big)^{a}\Gamma^2_{(a)}o_i\log\big(G(z_i)\big) + \frac{1}{B(a,b)}2\big(G(z_i)\big)^{a}\Gamma^2_{(a)}o_i\psi(a) - \frac{1}{B(a,b)}\big(G(z_i)\big)^{a}\Gamma^2_{(a)}o_i[\psi(a)-\psi(a+b)]d_i\Bigg[-\frac{1}{B(a,b)}\big(G(z_i)\big)^{a}\Gamma^2_{(a)}o_i + I_{(G(z_i))}(a,b)D_i - I_{(G(z_i))}(a,b)\big[-\psi^{(1)}_{(a)}+\psi^{(1)}_{(a+b)}\big]\Bigg] + \frac{1}{B(a,b)\big(G(z_i)\big)^{a}\Gamma^2_{(a)}[o_i]_a}\Bigg),$$

$$\zeta 2 = \Big(\big(1-G(z_i)\big)^{b}\Gamma^2_{(b)}v_i log\big(1-G(z_i)\big) + 2\big(1-G(z_i)\big)^{b}\Gamma^2_{(b)}v_i\psi(b) + \Big(-\big(1-G(z_i)\big)^{b}\Gamma^2_{(b)}v_i + B_{(1-G(z_i))}(a,b)\log\big(1-G(z_i)\big)\Big)d_i + B_{(1-G(z_i))}(a,b)\Big\{\psi^{(1)}_{(b)}-\psi^{(1)}_{(a+b)}\Big\}\Big) + \big(1-G(z_i)\big)^{b}\Gamma^2_{(b)}[v_i]_b,$$

$$\zeta 3 = \sum_{\mathcal{C}}\Big\{-\tfrac{1}{B(a,b)}\big(G(z_i)\big)^{a}\Gamma^2_{(a)}[o_i](\psi(b)-\psi(a+b)) - \tfrac{1}{B(a,b)}\Big(\big(1-G(z_i)\big)^{b}\Gamma^2_{(a)}v_i + B_{(1-G(z_i))}(a,b)d_i\Big)D_i - B_{(G(z_i))}(a,b)\psi^{(1)}_{(a+b)} - \tfrac{1}{B(a,b)}\big(G(z_i)\big)^{a}\Gamma^2_{(a)}[o_i]_a\Big\},$$

$$\zeta 4 = -\sum_{\mathcal{C}}\Bigg[-\frac{(-\xi 1_i)\ B_{(1-G(z_i))}(b,a)}{\big(1-G(z_i)\big)} + b\big(1-G(z_i)\big)^{b-1}(-\xi 1_i)\Gamma^2_{\mathrm{b}}\Big[\frac{1}{\Gamma(b+1)}u_i - v_i\Big] + b\big(1-G(z_i)\big)^{b-1}(-\xi 1_i)\Gamma^2_{\mathrm{b}}v_i + \big(G(z_i)\big)^{a-1}\big(1-G(z_i)\big)^{(b-1)}(-\xi 1_i)d_i\Bigg],$$

$$\zeta 5 = -\sum_{\mathcal{C}} e^{-z_i}(1+e^{z_i})(\xi 2_i)\big(I_{(G(z_i))}(a,b)\big) + \frac{1}{B(a,b)}ae^{-z_i}\big(G(z_i)\big)^{a}(1+e^{z_i})(\xi 2_i)\Gamma^2_{(a)}\Big(\frac{q_i}{\Gamma(1+a)} - o_i\Big) + \sum_{\mathcal{C}}\Big[\frac{1}{B(a,b)}a\big(G(z_i)\big)^{a-1}(\xi 2_i)\Gamma^2_{(a)}[o_i]\Big] - \sum_{\mathcal{C}}\Big[\frac{1}{B(a,b)}\big(G(z_i)\big)^{a-1}\big(1-G(z_i)\big)^{(b-1)}(\xi 2_i)D_i\Big],$$

$$z_i = \tfrac{y_i - x_i\beta}{\sigma},\ G(z_i) = \Big(\tfrac{e^{z_i}}{1+e^{z_i}}\Big)\ ,\ B(a,b) = \tfrac{\Gamma a\Gamma b}{\Gamma(a+b)} = \int_0^1 \varpi^{a}(1-\omega)^{b-1}d\varpi,$$

$$B_{(1-G(z_i))}(a,b) = \int_0^{(1-G(z_i))}\varpi^{a}(1-\omega)^{b-1}d\varpi,\ \ I_{(G(z_i))}(a,b) = \frac{1}{B(a,b)}\int_{(G(z_i))}^{1}\varpi^{a}(1-\omega)^{b-1}d\varpi,$$

$$D_i = \big(log\big(G(z_i)\big) - \psi(a) + \psi(a+b)\big), d_i = \Big(-\log\big(1-G(z_i)\big) + \psi(b) - \psi(a+b)\Big),$$

$$o_i = \ {}_3F_2\big(a,a,1-b;1+a,1+a;G(z_i)\big), [o_i]_a = \partial_a\big[\ {}_3F_2\big(a,a,(1-b);(1+a),(1+a);G(z_i)\big)\big],$$

$${}_3F_2\big(a,a,1-b;1+a,1+a;G(z_i)\big) = \sum_{n=0}^{\infty}\frac{(a)_n(a)_n(1-b)_n}{(1+a)_n(1+a)_n}\frac{\big(G(z_i)\big)^{n}}{n!},$$

$$v_i = \ {}_3F_2\Big(b,b,(a-1);(b+1),(b+1);\big(1-G(z_i)\big)\Big),$$

$$[v_i]_b = \partial_b \left[\ {}_3F_2\left(b, b, (a-1); (b+1), (b+1); \left(1 - G(z_i)\right)\right)\right],$$

$$q_i = \ {}_2F_1\left(a, (1-b); (1+a); G(z_i)\right), \ u_i = \ {}_2F_1\left(b, (1-a); (b+1); \left(1 - G(z_i)\right)\right),$$

$$\xi 1_i = \left(\frac{x_i e^{2z_i}}{\sigma[1+e^{z_i}]^2} - \frac{x_i e^{z_i}}{\sigma[1+e^{z_i}]}\right), \xi 2_i = \left(\frac{z_i e^{2z_i}}{\sigma[1+e^{z_i}]^2} - \frac{z_i e^{z_i}}{\sigma[1+e^{z_i}]}\right),$$

$$\zeta 1 = \left(\frac{1}{B(a,b)}\left(G(z_i)\right)^a \Gamma^2_{(a)} o_i \log\left(G(z_i)\right) + \frac{1}{B(a,b)} 2\left(G(z_i)\right)^a \Gamma^2_{(a)} o_i \psi(a) - \frac{1}{B(a,b)}\left(G(z_i)\right)^a \Gamma^2_{(a)} o_i [\psi(a) - \psi(a+b)] d_i \left[-\frac{1}{B(a,b)}\left(G(z_i)\right)^a \Gamma^2_{(a)} o_i + I_{(G(z_i))}(a,b) D_i - I_{(G(z_i))}(a,b)\left[-\psi^{(1)}_{(a)} + \psi^{(1)}_{(a+b)}\right]\right] + \frac{1}{B(a,b)\left(G(z_i)\right)^a \Gamma^2_{(a)}[o_i]_a}\right),$$

$$\begin{aligned}\zeta 2 = \Big(&\left(1 - G(z_i)\right)^b \Gamma^2_{(b)} v_i log\left(1 - G(z_i)\right) + 2\left(1 - G(z_i)\right)^b \Gamma^2_{(b)} v_i \psi(b) \\ &+ \left(-\left(1 - G(z_i)\right)^b \Gamma^2_{(b)} v_i + B_{(1-G(z_i))}(a,b) \log\left(1 - G(z_i)\right)\right) d_i \\ &+ B_{(1-G(z_i))}(a,b)\left\{\psi^{(1)}_{(b)} - \psi^{(1)}_{(a+b)}\right\}\Big) + \left(1 - G(z_i)\right)^b \Gamma^2_{(b)} [v_i]_b,\end{aligned}$$

$$\zeta 3 = \sum_{\mathcal{C}} \left\{-\frac{1}{B(a,b)}\left(G(z_i)\right)^a \Gamma^2_{(a)}[o_i](\psi(b) - \psi(a+b)) - \frac{1}{B(a,b)}\left(\left(1 - G(z_i)\right)^b \Gamma^2_{(a)} v_i + B_{(1-G(z_i))}(a,b) d_i\right) D_i - B_{(G(z_i))}(a,b)\psi^{(1)}_{(a+b)} - \frac{1}{B(a,b)}\left(G(z_i)\right)^a \Gamma^2_{(a)}[o_i]_a\right\},$$

$$\begin{aligned}\zeta 4 = -\sum_{\mathcal{C}} \Bigg[&-\frac{(-\xi 1_i)\ B_{(1-G(z_i))}(b,a)}{\left(1 - G(z_i)\right)} + b\left(1 - G(z_i)\right)^{b-1}(-\xi 1_i)\Gamma^2_{\mathrm{b}}\left[\frac{1}{\Gamma(b+1)} u_i - v_i\right] \\ &+ b\left(1 - G(z_i)\right)^{b-1}(-\xi 1_i)\Gamma^2_{\mathrm{b}} v_i + \left(G(z_i)\right)^{a-1}\left(1 - G(z_i)\right)^{(b-1)}(-\xi 1_i) d_i\Bigg],\end{aligned}$$

and

$$\begin{aligned}\zeta 5 = -\sum_{\mathcal{C}} &e^{-z_i}(1+e^{z_i})(\xi 2_i)\left(I_{(G(z_i))}(a,b)\right) + \frac{1}{B(a,b)} a e^{-z_i}\left(G(z_i)\right)^a (1+e^{z_i})(\xi 2_i)\Gamma^2_{(a)}\left(\frac{q_i}{\Gamma(1+a)} - o_i\right) \\ &+ \sum_{\mathcal{C}}\left[\frac{1}{B(a,b)} a\left(G(z_i)\right)^{a-1}(\xi 2_i)\Gamma^2_{(a)}[o_i]\right] \\ &- \sum_{\mathcal{C}}\left[\frac{1}{B(a,b)}\left(G(z_i)\right)^{a-1}\left(1 - G(z_i)\right)^{(b-1)}(\xi 2_i) D_i\right].\end{aligned}$$