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# The Non-Negative $P_0$ –Matrix Completion Problem for 5x5 Matrices Specifying Digraphs with 5 Vertices and 3 Arcs.

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**Abstract**

The non-negative $P_0$ – matrix completion is considered for 5x5 matrices specifying digraphs for p = 5, q = 3, where p is number of vertices and q is number of arcs by performing zero completion on the matrices. The study establishes that all digraphs for p = 5, q = 3 specifying 5x5 partial matrices which are either cycles or acyclic digraphs have non-negative $P_0$ –completion.

***Keywords:*** Principal submatrix, partial matrix, matrix completion, $P_0$ –matrix, Nonnegative $P_0$ –matrix

## 1. Introduction

A matrix *A* is a rectangular array of numbers or objects arranged in rows and columns. A submatrix of a matrix *A* is a smaller matrix obtained by deleting a collection of row(s) and / or column(s) from the matrix *A*. If *A* is an nxn matrix, for α subset of {1, 2, ----, n}, the **principal submatrix** *A*(α) is obtained by deleting all rows and columns that are not in α. A principal minor is the determinant of a leading principal submatrix obtained by deleting the last n-k rows and n-k columns of the nxn matrix *A*. for nxn square matrix there are n leading principal minors.

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A **partial matrix** is a matrix in which some entries are specified while the remaining unspecified entries are free to be chosen [5]. Example $A = \begin{bmatrix} 4 & 2 & x \\ 2 & 1 & y \\ 4 & -1 & 1 \end{bmatrix}$ is a 3x3 partial matrix with elements in positions (1,1), (1,2), (2,1), (2,2), (3, 1), (3,2), (3,3) specified while elements in positions (1,3), (2,3) are unspecified. A fully specified principal submatrix such as $A(1,2)$ of matrix *A* above has all entries specified. Completion of a partial matrix is a particular choice of values for unspecified entries so that the resulting matrix specifies a certain property.

An nxn matrix has a list of positions given by {1, 2, -------, n} x {1, 2, -------, n}. If Q is a subset of this list of positions, then Q is said to be pattern of the nxn matrix. A partial matrix specifies the pattern if its specified entries are those exactly listed in the pattern. For instance the partial matrix *A* above specifies the pattern {(1,1), (1,2), (2,1), (2,2), (3.1), (3,2), (3,3)}.

Matrices are of various classes such as positive definite, P, $P_0$, $M_0$, nonnegative $P_0$matrices and others. Each of the class specifies certain properties. As stated in[1], for a particular class $\prod$ of matrices, a pattern is said to have $\prod$- completion if every partial $\prod$- matrix specifying the pattern can be completed to a $\prod$- matrix. If there exists even one partial $\prod$- matrix specifying the pattern that cannot be completed that pattern is said not to have completion.

A real nxn matrix is called **a $P_0$- matrix** if all its principal minors are non-negative. A partial $P_0$-matrix is a partial matrix in which all fully specified sub-matrices are $P_0$- matrices. A real nxn matrix is **nonnegative $P_0$- matrix** if all entries are nonnegative and all its principal minors are nonnegative i.e. it's a $P_0$- matrix whose entries are nonnegative. A partial matrix is a partial nonnegative $P_0$- matrix if determinants of all fully specified sub-matrices are nonnegative and all specified entries are nonnegative. A pattern is said to have a nonnegative $P_0$- completion if every partial nonnegative $P_0$ – matrix specifying the pattern can be completed to a nonnegative $P_0$-matrix. These are well defined in [4].

Graphs and digraphs have been used effectively to study matrix completion problems. For positionally symmetric pattern Q that includes all diagonal positions, the graph of Q (pattern graph) is used to carry out the study. For patterns without positional symmetry, digraphs (directed graphs) are used, as established in [5].

Digraphs assist the study of nonnegative $P_0$- matrix completion since the case considered involve patterns involving 5x5 matrices with all diagonal entries specified and not necessarily for position {j, i } to be in the pattern if position {i, j}is in the pattern.

**A digraph** is ordered pair D = (V, A) comprising of a set of vertices together with a set A of directed edges called arcs. The order of a digraph is the number of vertices in the digraph while the size of a digraph is the number of arcs in the digraph. A digraph H is said to be a sub-digraph of D if every vertex of H is also a vertex of D and every arc of H is also an arc of D. [6]

Let D be a digraph, a path that begins and ends at the same vertex is called a cycle. A digraph that does not contain any cycles is called an acyclic digraph. A chord is an arc joining two non-consecutive vertices of a cycle. A digraph is chordal if any cycle of length > 3 has a chord. A subset of a directed graph is called a clique if it contains at least three vertices and for each pair of vertices $v_i$ and $v_j$ in the subset, both $v_i \rightarrow v_j$ and $v_j \rightarrow v_i$ are true. [4]

In many situations it is convenient to permute entries of a partial matrix. A permutation matrix P is obtained by interchanging rows on the identity matrix. The permutation matrix A is then $PAP^T$. This is represented on the digraph by renumbering the vertices. As a result of the following lemma we are allowed to permute a partial nonnegative $P_0$ –matrix and hence renumber digraph vertices as convenient.

**Lemma 1.1** [1]: *The class of nonnegative $P_0$ –matrices is closed under permutation.*

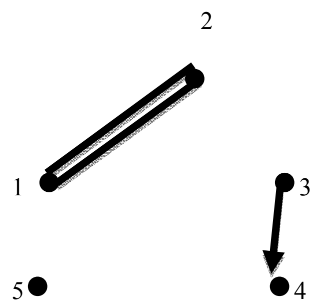
Some studies have been done on nonnegative $P_0$ –matrix completion. In [4], Hogben established that for nonnegative $P_0$ -matrices, patterns of every non-separable strongly connected induced sub-digraph has nonnegative $P_0$- completion. In the same study it is shown that all 3x3 matrices have nonnegative $P_0$ –completion prove of which is given in [2]. In [5], Hogben established that a pattern that has nonnegative $P_0$ -completion also have nonnegative P- completion. In [2], it is established that a 4x4 matrix that includes all diagonal positions has nonnegative $P_0$ –completion if and only if its digraph is complete when it has a 4-cycle. Also shown in the study is that any positionally symmetric pattern that includes all diagonal positions and whose graph is an n-cycle has nonnegative $P_0$ –completion if and only if $n \neq 4$.

In next section all possible digraphs with 5 vertices and 3 arcs are considered and 5x5 partial matrices specifying the digraphs extracted. The construction of digraphs will be with the guidance of graphs with five points and three lines as given in [3]

The process of extracting the partial nonnegative $P_0$ –matrices will be as follows: A specific entry $a_{ij}$ will be used to represent the corresponding present arc in the digraph, an unspecified entry $x_{ij}$ will represent a corresponding missing arc in the digraph while $d_{ii}$ will specify the diagonal entries. Zero completion method will then be used to find out whether each of the cases have zero completion to nonnegative $P_0$ -matrix or not.

**Classification of 5x5 matrices specifying digraphs with 5 vertices and 3 arcs.**

i) **Consider the digraph below:**



Let $A = \begin{pmatrix} d_{11} a_{12} x_{13} x_{14} x_{15} \\ a_{21} d_{22} x_{23} x_{24} x_{25} \\ x_{31} x_{32} d_{33} a_{34} x_{35} \\ x_{41} x_{42} x_{43} d_{44} x_{45} \\ x_{51} x_{52} x_{53} x_{54} d_{55} \end{pmatrix}$ be a partial nonnegative $P_0$-matrix representing the digraph above.

Determining the determinants of all the principal minors then setting the unspecified entries to zero, i.e. $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{23} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. Determinants of the principal sub-matrices will be as follows:-

Det $A(1,2) = d_{11}d_{22} - a_{12}a_{21} \geq 0$ since $A(1,2)$ is fully specified.

Similarly, Det $A(1,3)$, Det $A(1,4)$, Det $A(1,5)$, Det $A(2,3)$, Det $A(2,4)$, Det $A(2,5)$, Det $A(3,5)$, Det $A(4,5) \geq 0$

Det $A(1,2,3) = d_{11}d_{22}d_{33} - a_{12}a_{21}d_{33} = d_{33}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified.

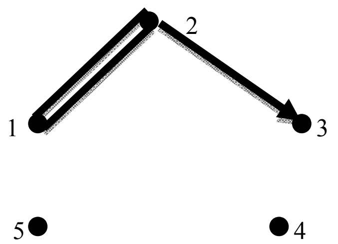
Similarly Det $A(1,2,4)$, Det $A(1,2,5)$, Det $A(1,3,4)$, Det $A(1,3,5)$, Det $A(1,4,5)$, Det $A(2,3,4)$, Det $A(2,3,5)$, Det $A(2,4,5)$, Det $A(3,4,5) \geq 0$

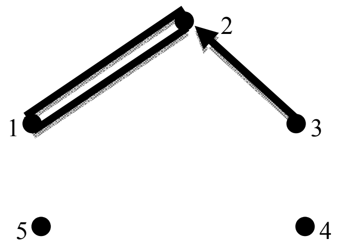
Det $A(1,2,3,4) = d_{11}d_{22}d_{33}d_{44} - a_{12}a_{21}d_{33}d_{44} = d_{33}d_{44}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified

Similarly, Det $A(1,2,3,5)$, Det $A(1,2,4,5)$, Det $A(1,3,4,5)$, Det $A(2,3,4,5) \geq 0$

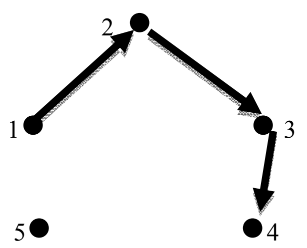
Det $A = d_{11}d_{22}d_{33}d_{44}d_{55} - a_{12}a_{21}d_{33}d_{44}d_{55} = d_{33}d_{44}d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$, since $A(1,2)$ is fully specified

Hence all principal minors are nonnegative and therefore partial matrix has zero completion into nonnegative $P_0$ –matrix. Carrying out similar procedures for the following digraphs similar results will be obtained.




**ii) Consider the digraph below:**



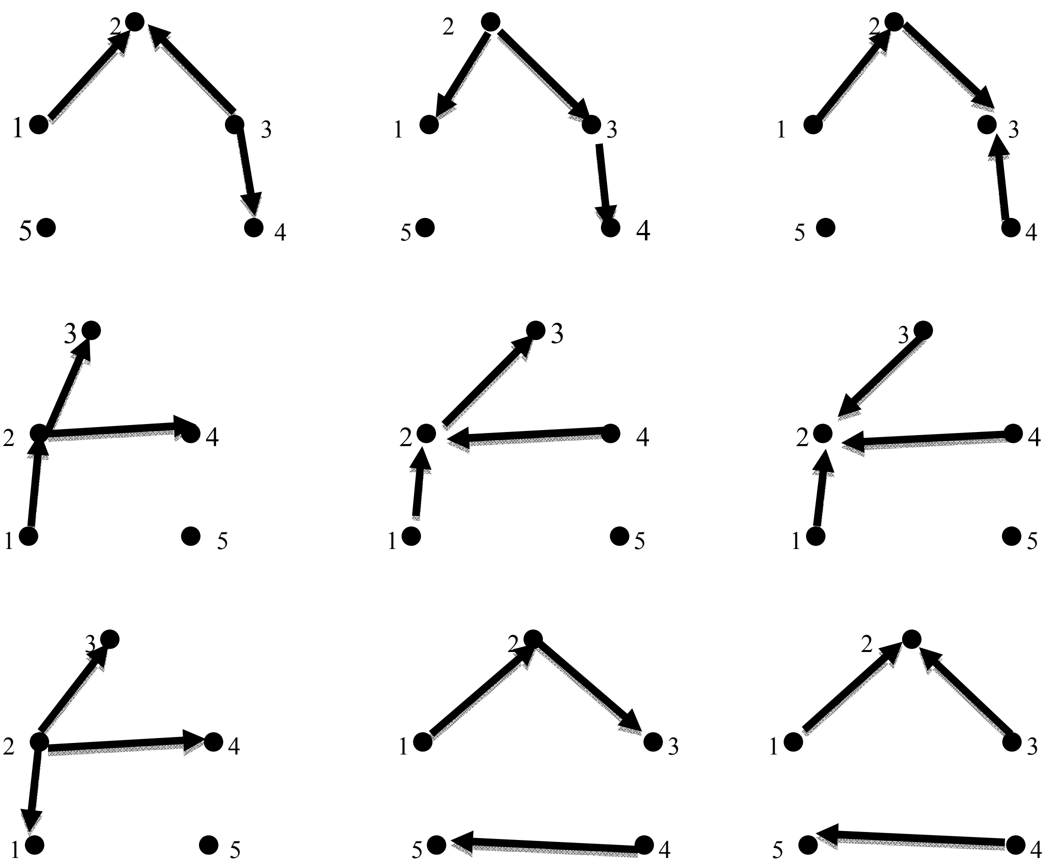
Let $A = \begin{pmatrix} d_{11} a_{12} x_{13} x_{14} x_{15} \\ x_{21} d_{22} a_{23} x_{24} x_{25} \\ x_{31} x_{32} d_{33} a_{34} x_{35} \\ x_{41} x_{42} x_{43} d_{44} x_{45} \\ x_{51} x_{52} x_{53} x_{54} d_{55} \end{pmatrix}$ be a partial nonnegative $P_0$-matrix representing the digraph above.

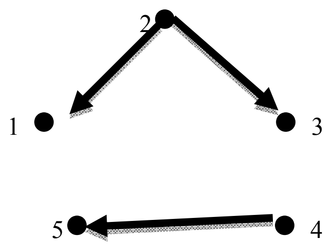
Determining the determinants of all the principal submatrices, then setting the unspecified entries to zero, i.e. $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{23} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. Determinants of the principal sub-matrices will be follows:

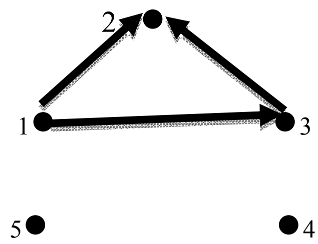
Det $A(1,2)$, Det $A(1,3)$, Det $A(1,4)$, Det $A(1,5)$, Det $A(2,3)$, Det $A(2,4)$, Det $A(2,5)$, Det $A(3,5)$, Det $A(4,5)$, Det $A(1,2,3)$, Det $A(1,2,4)$, Det $A(1,2,5)$, Det $A(1,3,4)$, Det $A(1,3,5)$, Det $A(1,4,5)$, Det $A(2,3,4)$, Det $A(2,3,5)$, Det $A(2,4,5)$, Det $A(3,4,5)$, Det $A(1,2,3,4)$, Det $A(1,2,3,5)$, Det $A(1,2,4,5)$, Det $A(1,3,4,5)$, Det $A(2,3,4,5)$, Det $A \geq 0$

Hence all principal minors are nonnegative and therefore partial matrix has zero completion into nonnegative $P_0$ –matrix.

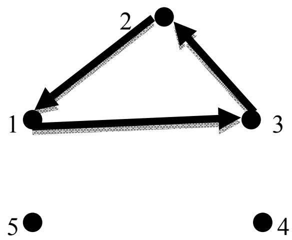
Carrying out similar procedures to all other acyclic digraphs shown below, similar results will be obtained.






## iii) Consider the following digraph which is a cycle




Let $A = \begin{pmatrix} d_{11} x_{12} a_{13} x_{14} x_{15} \\ a_{21} d_{22} x_{23} x_{24} x_{25} \\ x_{31} a_{32} d_{33} x_{34} x_{35} \\ x_{41} x_{42} x_{43} d_{44} x_{45} \\ x_{51} x_{52} x_{53} x_{54} d_{55} \end{pmatrix}$ be a partial nonnegative $P_0$-matrix representing the digraph above.

Determining the determinants of all the principal minors then setting the unspecified entries to zero, i.e. $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{23} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{35} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{45} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$. Determinants of the principal sub-matrices can be shown as above to be $\geq 0$

Hence all principal minors are nonnegative and therefore partial matrix has zero completion into nonnegative $P_0$ –matrix.

**Conclusion**

All the digraphs for 5x5 matrices with 3 arcs which are either cycles or acyclic digraphs have zero completion into nonnegative $P_0$ –matrix**.**

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