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Theoretical Study of Angular Momentum in Quantum Mechanics

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Abstract

Angular momentum can simply be interpreted as the momentum of an object that rotates or rotates. Angular momentum or angular center intuitively measures how much linear momentum is directed around a certain point or often called the center point. In this paper, an analysis of angular momentum in quantum mechanics has been carried out, as well as a study of the expansion of the application of angular momentum in quantum mechanics in the form of the Clebsch–Gordan coefficient.

Keywords: angular momentum; quantum mechanics.

1. Introduction

In physics, angular momentum is important because angular momentum is a physical quantity that is always relevant when studying atoms. Besides that, angular momentum has a close relationship also in studying moments of inertia and torque. Previously, angular momentum was given in modern form via Noether's theorem. The conservation of angular momentum is discussed in two ways, related to the inertia of rotation of objects and the motion of planet revolutions [1]. Angular momentum in classical mechanics is given in vector form. The plane perpendicular to this vector, according to the central field theory, determines the space in which the movement of the particles takes place. This cannot be studied simply in quantum mechanics.

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The state of the particle at the center of the field is proportional to the spherical harmonics, which do not define any plane of motion. In classical mechanics vectors are added, whereas in quantum mechanics the Clebsch–Gordan coefficient must be used. A classical approach to the quantum coefficients and the limits of its application have been found. This analysis provides a basis for the vector addition model used in several basic studies of atomic physics. This can help to better understand the addition of angular momentum in quantum mechanics [2]. In the Schrodinger equation which appears in coupling treatments in quantum mechanics such as atomic collisions which involve fine structure effects, alternative representations are developed in angular momentum algebra. The various representations are closely related to Hund's coupling scheme for rotating diatomic molecules. Matrix elements for electrostatic interactions and for orthogonal transformations connecting the various representations, are given explicitly for the case when only one atom has internal angular momentum and is subject to LS coupling [3]. Canonical angular momentum of a free electron, positron and gamma photon [4]. From the expression above, it can be seen that angular momentum in quantum mechanics is still interesting to study, especially the appearance in the operator form, and how it results in the sum of angular momentum, how the eigenvalues of angular momentum are formed. So the goal of this paper is to study in more detail the angular momentum in quantum mechanics and to analyze the Clebsh-Gordan coefficient.

2. Method

The method used in this study is a theoretical method by theoretically analyzing various existing literature and various references, then describing the angular momentum analytic form and continuing to analyze the Clebsh-Gordan coefficient.

3. Results and Discussion

Angular momentum

Angular momentum can simply be interpreted as the momentum of an object that rotates or rotates. Angular momentum or angular center intuitively measures how much linear momentum is directed around a certain point or often called the center point. In classical mechanics, the angular momentum of a particle is given by the equation.

$$L = r\ x\ p, \tag{1}$$

To obtain the cross product of the radius of the rotating axis (*r*) with the linear momentum (*p*), you can use the determinant matrix as follows:

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} \tag{2}$$

From equation (2) three equations of angular momentum on the rotary axis (r) are obtained as follows:

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \text{ dan } L_z = xp_y - yp_x, \tag{3a}$$

The momentum operator for cartesian coordinates is obtained:

$$\hat{p} = -i\hbar\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \tag{3b}$$

Then substitute equation (3b) into equation (1):

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ -i\hbar\frac{\partial}{\partial x} & -i\hbar\frac{\partial}{\partial y} & -i\hbar\frac{\partial}{\partial z} \end{vmatrix} \tag{4}$$

The corresponding quantum operators obtained from equation (4) are as follows [5]:

$$L_x = y\left(-i\hbar\frac{d}{dz}\right) - z\left(-i\hbar\frac{d}{dy}\right) = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \tag{5}$$

$$L_y = z\left(-i\hbar\frac{d}{dx}\right) - x\left(-i\hbar\frac{d}{dz}\right) = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \tag{6}$$

$$L_x = x\left(-i\hbar\frac{d}{dy}\right) - y\left(-i\hbar\frac{d}{dx}\right) = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{7}$$

Notes: $ix - i = 1$

$$-i = \frac{1}{i}$$

Then we get the cartesian coordinate momentum operator as follows:

$$\hat{L} = -i\hbar\left[\hat{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \hat{j}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) + \hat{k}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right] \tag{8}$$

After obtaining the cartesian coordinate momentum operator, we then look for the momentum operator in spherical coordinates as follows:

$$\hat{p} = -i\hbar\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \tag{9}$$

Then substitute equation (9) into equation (1) :

$$\begin{aligned}
\hat{L} &= (r\hat{r})x\hat{p} \\
\hat{L} &= -i\hbar r\hat{r}x\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \\
\hat{L} &= -i\hbar\left(r(\hat{r}x\hat{r})\frac{\partial}{\partial r} + (\hat{r}x\hat{\theta})\frac{\partial}{\partial \theta} + (\hat{r}x\hat{\phi})\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right)
\end{aligned} \tag{10}$$

Noted that $\hat{r}x\hat{r} = 0$, $\hat{r}x\hat{\theta} = \hat{\phi}$ dan $\hat{r}x\hat{\phi} = -\theta$, then we get the equation (10) become :

$$\hat{L} = -i\hbar\left(\hat{\phi}\frac{\partial}{\partial \theta} - \theta\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right) \tag{11}$$

In the previous section it is known that $\hat{\theta}$ and $\hat{\phi}$ the equation is obtained

$$\hat{\theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k} \tag{12}$$

$$\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j} \tag{13}$$

Then substitute equation (12) and equation (13) into equation (10):

$$\begin{aligned}
\hat{L} &= -i\hbar\left(\hat{\phi}\frac{\partial}{\partial \theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right) \\
\hat{L} &= -i\hbar\left[\left(\hat{i} - \sin\phi + \hat{j}\cos\phi\right)\frac{\partial}{\partial \theta} - \left(\hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta\right)\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right] \\
\hat{L} &= -i\hbar\left[\left(\hat{i} - \sin\phi\frac{\partial}{\partial \theta} + \hat{j}\cos\phi\frac{\partial}{\partial \theta}\right) - \begin{pmatrix}\hat{i}\cos\theta\cos\phi\frac{1}{\sin\theta}\frac{\partial}{\partial \phi} + \hat{j}\cos\theta\sin\phi \\ \frac{1}{\sin\theta}\frac{\partial}{\partial \phi} - \hat{k}\sin\theta\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\end{pmatrix}\right] \\
\hat{L} &= -i\hbar\left[\left(\hat{i} - \sin\frac{\phi\partial}{\partial \theta} + \hat{j}\cos\frac{\phi\partial}{\partial \theta}\right) - \left(\hat{i}\frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial \phi} + \hat{j}\frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial \phi} - \hat{k}\frac{\sin\theta}{\sin\theta}\frac{\partial}{\partial \phi}\right)\right] \\
\hat{L} &= -i\hbar\left[\left(\hat{i} - \sin\frac{\phi\partial}{\partial \theta} + \hat{j}\cos\frac{\phi\partial}{\partial \theta}\right) - \left(\hat{i}\cot\phi\cos\phi\frac{\partial}{\partial \phi} + \hat{j}\cot\phi\sin\phi\frac{\partial}{\partial \phi} - \hat{k}.1.\frac{\partial}{\partial \phi}\right)\right]
\end{aligned}$$

$$\hat{L} = -i\hbar i\left[\hat{i} - \sin\frac{\phi\partial}{\partial\theta} - \hat{i}\cot\theta\cos\frac{\phi\partial}{\partial\theta} + \hat{j}\cos\phi\frac{\partial}{\partial\phi} - \hat{j}\cot\phi\sin\phi\frac{\partial}{\partial\phi} + \hat{k}\frac{\partial}{\partial\phi}\right]$$

$$\hat{L} = -i\hbar i\left[\hat{i}\left(-\sin\frac{\phi\partial}{\partial\theta} - \cot\phi\cos\phi\frac{\partial}{\partial\phi}\right) + \hat{j}\left(\cos\phi\frac{\partial}{\partial\phi} - \cot\phi\sin\phi\frac{\partial}{\partial\phi}\right) + \hat{k}\frac{\partial}{\partial\phi}\right]$$

$$\hat{L} = -i\hbar i\begin{bmatrix}\hat{i}\left(-\sin\frac{\phi\partial}{\partial\theta} - \cot\phi\cos\phi\frac{\partial}{\partial\phi}\right) + \\ \hat{j}\left(\cos\phi\frac{\partial}{\partial\phi} - \cot\phi\sin\phi\frac{\partial}{\partial\phi}\right) + \hat{k}\left(\frac{\partial}{\partial\phi}\right)\end{bmatrix} \tag{14}$$

Notes: $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

The spherical coordinate angular momentum operator is obtained as follows

$$L_x = -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right) \tag{15}$$

$$L_y = -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right) \tag{16}$$

$$L_x = -i\hbar\frac{\partial}{\partial\phi} \tag{17}$$

After the operator L_x, L_y and L_z is obtained, then the eigenvalues and eigenfunctions can be searched [6].

Eigen Value

L_x and L_y is not alternating (non-commutative), where

$$[L_x, L_y] = L_x L_y - L_x L_y \neq 0 \tag{18}$$

In analyzing $[L_x, L_y]$ operated test function *f(x, y, z)* to $[L_x, L_y]$:

$$[L_x,L_y]f=\left(\frac{\hbar}{i}\right)^z\left\{\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)f-\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)f\right\}$$

$$[L_x,L_y]f=\left(\frac{\hbar}{i}\right)^2\left\{\begin{matrix}y\frac{\partial}{\partial z}\left(z\frac{\partial f}{\partial x}\right)-y\frac{\partial}{\partial z}\left(x\frac{\partial f}{\partial z}\right)-z\frac{\partial}{\partial y}\left(y\frac{\partial f}{\partial y}\right)+z\frac{\partial}{\partial x}\left(z\frac{\partial f}{\partial y}\right)\\-x\frac{\partial}{\partial z}\left(z\frac{\partial f}{\partial y}\right)+z\frac{\partial}{\partial x}\left(z\frac{\partial f}{\partial y}\right)+x\frac{\partial}{\partial z}\left(y\frac{\partial f}{\partial z}\right)-x\frac{\partial}{\partial z}\left(z\frac{\partial f}{\partial y}\right)\end{matrix}\right\}$$

$$[L_x,L_y]f=\left(\frac{\hbar}{i}\right)^2\begin{pmatrix}y\frac{\partial f}{\partial x}+yz\frac{\partial^2 f}{\partial z\partial x}-yx\frac{\partial^2 f}{\partial z^2}-z^2\frac{\partial^2 f}{\partial y\partial x}+zx\frac{\partial^2 f}{\partial y\partial z}\\-zy\frac{\partial^2 f}{\partial x\partial z}+z^2\frac{\partial^2 f}{\partial x\partial y}+xy\frac{\partial^2 f}{\partial z^z}-x\frac{\partial f}{\partial y}-xz\frac{\partial^2 f}{\partial z\partial y}\end{pmatrix}$$

All paired terms (based on cross-derivative equations) except two:

$$[L_x,L_y]f=\left(\frac{\hbar}{i}\right)^2\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)f=i\hbar L_z f$$

and because $L_z=i\hbar\left[y\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial y}\right]$ can simplify the final result to be

$$[L_x,L_y]f=i\hbar L_z f \tag{19}$$

Then obtained:

$$[L_x,L_y]=-[L_x,L_y]=i\hbar L_z \tag{20}$$

with cyclical permutations of indices, it also follows that

$$[L_y,L_z]=i\hbar L_x \quad \text{dan} \quad [L_z,L_x]=i\hbar L_y \tag{21}$$

From this fundamental commutation relationship, the entire theory of angular momentum can be deduced. Proven L_x, L_y and L_z, is an incompatible observable. According to the general uncertainty principle:
$\sigma_{L_x}^2\sigma_{L_y}^2\geq\left(\frac{1}{2i}\langle i\hbar L_z\rangle\right)^2=\frac{\hbar^2}{4}\langle L_z\rangle^2$

or

$$\sigma_{L_x}\sigma_{L_y}\geq\frac{\hbar}{4}\left|\langle L_z\rangle\right| \tag{22}$$

Therefore, we can look for the commutative relationship between L^2 and $L_\pm$ obtained as follows

$$[L^2, L_\pm] = L^2 L_\pm - L_\pm L^2 \tag{23}$$

If $[L^2, L_\pm] = L^2 L_\pm - L_\pm L^2 = 0$, so it can be obtained that the relationship between the two is commutative and $L^2 L_\pm = L_\pm L^2$

To find conditions that are simultaneously eigenfunctions of *Lx* and of *Ly*. On the other hand, the square of the total angular momentum is obtained:

$$L^2 \equiv L_x^2 + L_y^2 + L_z^2 \tag{24}$$

Do a test with L_x:

$$[L^2, L_x] = [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x]$$
$$[L^2, L_x] = L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_y, L_x] + [L_y, L_x]L_z$$
$$[L^2, L_x] = L_y(-i\hbar L_z) + (-i\hbar L_z)L_y + L_z(i\hbar L_y) + (i\hbar L_z)L_z$$
$$[L^2, L_x] = 0$$

Followed by L^2 also tested with *Lx* and *Ly*

$$[L^2, L_x] = 0 \text{ , } [L^2, L_y] = 0 , \tag{25}$$

Or

$$[L^2, L\ \ ] = 0 \tag{26}$$

So $L^2$2 is compatible with every component of *L*, and one can expect to find simultaneous eigenstates of L^2 and *Lz*

$$L^2 f = \lambda f \text{ and } L_z f = \mu f \tag{27}$$

We will then use the ladder operator technique, very similar to that applied to the harmonic oscillator

$$L_\pm \equiv L_x \pm iL_y \tag{28}$$

Its commutator with L_z is

$$[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y] = i\hbar L_y \pm i(-i\hbar L_x) = \pm\hbar(L_x \pm iL_y)$$

So that

$$[L_z, L_\pm] = \pm\hbar L_\pm \tag{29}$$

and

$$[L^2, L_\pm] = 0 \tag{30}$$

That if f is an eigen function of L^2 and L_z, also is $L_\pm f$. For Equation (29) obtained:

$$L^2(L_\pm f) = L_\pm(L^2 f) = L_\pm(\lambda f) = \lambda(L_\pm f) \tag{31}$$

so$L_\pm f$ is the eigenfunction ofL^2, with the same eigenvalues λ, and Equation (29)

$$\begin{aligned} L_z(L_\pm f) &= (L_\pm L_z - L_z L_\pm) f + L_\pm L_z f = \pm\hbar L_\pm f + L_\pm(\mu f) \\ L_z(L_\pm f) &= (\mu \pm \hbar)(L_\pm f) \end{aligned} \tag{32}$$

So $L_\pm f$ is the eigenfunction of L_z. with new eigenvalues $\mu \pm \hbar$. L_+ is called the "increasing" operator because it increases the eigenvalue L_z, by $\hbar$, and L- is called the "decreasing" operator because it decreases the eigenvalue by $\hbar$ [6].

One constructs finite-dimensional, irreducible representations of the Lie algebra of the rotation group; from these, as we know, the representations of the local group follow by exponentiation. To each finite-dimensional irreducible representation there belongs a finite-dimensional irreducible invariant subspace; the basic states spanning this irreducible subspace are the angular momentum eigenstates: under rotations they are transformed among themselves (i.e. within that subspace) and the corresponding transformation matrices make up just the irreducible representation which leaves this subspace invariant [7].

For a given value λ, then, a "ladder" is obtained, with each "rung" separated from its neighbors by $\hbar$ one unit in the eigenvalues L_z

$$L_\pm f_t = 0 \tag{33}$$

$\hbar l$ being the eigenvalues of L_z, on this top rung (the corresponding letter l is sometimes called the azimuth quantum number will appear again):

$$L_z f_t = \hbar l f \;\; ; \;\; L^2 f_t = \lambda f_t \tag{34}$$

Now it becomes [8](Shankar, 1998):

$$L_{\pm}L_{\pm} = (L_{\pm}iL_{Y})(L_{x} \mp iL_{y}) = L_x^2 + L_y^2 \mp i(L_xL_y - L_yL_x)$$
$$L_{\pm}L_{\pm} = L^2 - L_z^2 \pm i(i\hbar L_z)$$

or vice versa

$$L^2 = L_{\pm}L_{\mp} + L_z^2 \mp \hbar L_z \quad (35)$$

therefore

$$L\ f_t = (L - L_+ + L_z^2 + \hbar L_z)f_t = \{0 + \hbar^2 l^2 + \hbar^2 l\} f_t = \hbar^2 l(l+1) f_t$$

and thus obtained:

$$\lambda = \hbar^2 l(l+1) \quad (36)$$

These are the eigenvalues of L^2 in terms of the maximum eigenvalues of L_z. Meanwhile, there is also the bottom rung, so f_b

$$L - f_b, =0. \quad (37)$$

Suppose $\hbar \bar{l}$ is the eigenvalue of *Lz*, on the lowest rung of this ladder

$$L_z f_b = \hbar \bar{l} f_b; \quad L^2 f_b = \lambda f_b; \quad (38)$$

by using Equation (35), is obtained

$$L^2 f_b = (L_+L_- + L_z^2 - \hbar L_z) f_b = \{0 + \hbar^2 \bar{l}^2 - \hbar^2 \bar{l}\} f_b = \hbar^2 \bar{l}(\bar{l} - 1) f_b$$

So that

$$\lambda = \hbar^2 \bar{l}(\bar{l} - 1) \quad (39)$$

Comparing Equation (38) and equation (39), it is seen that $l(l+1) = \bar{l}(\bar{l} - 1)$, so $\bar{l} = l + 1$ (which makes no sense is the lower rung is higher than the top rung), or else

$$\bar{l} = -1. \quad (40)$$

It is shown that the eigenvalue of L_z is $m\hbar$, where m (the correspondence of this letter will also be clear soon) starts from –*l* to *l* in *N* integer steps. Specifically, here *l* = -*l* + *N*, and therefore $l = \frac{N}{2}$, so *l* must be an integer or a half integer. The eigenfunctions are characterized by the numbers *l* and *m*

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m; \quad L_z f_l^m = \hbar m f_l^m \tag{41}$$

where

$$l = 0, 1/2, 1, 3/2, \ldots; \qquad m = -l, -1+1, \ldots, l-1, l. \tag{42}$$

For a given value of *l*, there are 2*l* + 1 different values of *m* (i.e., 2*l* + 1 "rung" in "Jadder"). In a purely algebraic way, starting with the fundamental commutation relations, we have determined the eigenvalues of L^2 and L_z without ever looking at the eigenfunctions themselves! Now turning to the problem of constructing the eigenfunctions. The point before we start $f_1^m = Y_1^m$: the eigenfunctions of L^2 and L_z, are nothing but.

Clebsh-Gordan coefficients

The operators $\left(\hat{J}^{(1)}\right)^2, \hat{J}_3^{(1)}, \left(\hat{J}^{(2)}\right)^2$ and $\hat{J}_3^{(2)}$ commute pairwise. Their simultaneous eigenvectors are

$$\left| j_1 j_2 m_1 m_2 \right\rangle := \left| j_1 m_1 \right\rangle \otimes \left| j_2 m_2 \right\rangle. \tag{43}$$

Similarly, $\left(\hat{J}^{(1)}\right)^2, \hat{J}_3^{(1)}, \left(\hat{J}^{(2)}\right)^2$ and $\hat{J}_3^{(2)}$ commute pairwise, too. We will construct vectors

$$\left| j_1 j_2 jm \right\rangle \tag{44}$$

Which are simultaneous eigenvectors, i.e., which satisfy

$$\left(\hat{J}^{(1)}\right)^2 \left| j_1 j_2 jm \right\rangle = \hbar^2 j_1 (j_1 + 1) \left| j_1 j_2 jm \right\rangle,$$

$$\left(\hat{J}^{(2)}\right)^2 \left| j_1 j_2 jm \right\rangle = \hbar^2 j_2 (j_2 + 1) \left| j_1 j_2 jm \right\rangle,$$

$$\hat{J}^2 \left| j_1 j_2 jm \right\rangle = \hbar^2 j(j+1) \left| j_1 j_2 jm \right\rangle,$$

$\hat{J}_3 | j_1 j_2 jm\rangle = \hbar m | j_1 j_2 jm\rangle$.

Since $\hat{J}_3 = \hat{J}_3^{(1)} + \hat{J}_3^{(2)}$, the $| j_1 j_2 jm\rangle$ are also eigenvectors of $\hat{J}_3$:

$$\hat{J}_3 | j_1 j_2 jm\rangle = \hbar(m_1 + m_2) | j_1 j_2 m_1 m_2\rangle$$

It follows that

$$m = m_1 + m_2 \tag{45}$$

Definition 1. *The expansions coefficients*

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 j\rangle$$

Of the vectors $| j_1 j_2 jm\rangle$ *in the basis* $| j_1 j_2 m_1 m_2\rangle$ *are called Clebsh-Gordan coefficients.*

Example 2. Consider the special case of two spin $\frac{1}{2}$ particles. In view of dealing with spins, we denote $s \equiv j$ and $\hat{S}_k \equiv \hat{J}_k$. The total spin operators is given by

$$\hat{S}_k = i\hbar\left(L^{1/2} \otimes L^{1/2}\right)(\hat{s}_k) \equiv \hat{S}^{(1)} + \hat{S}^{(2)}$$

We are going to construct the vectors $|s_1 s_2 sm\rangle$ explicity. According to(45),

$$m = 1,0,0,-1$$

Write down all vectors $|s_1 s_2 m_1 m_2\rangle$ and $|s_1 s_2 sm\rangle$ and given them a shorthand notation:

$\lvert s_1 s_2 m_1 m_2\rangle$	$\lvert s_1 s_2 sm\rangle$
$\left\lvert \frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\right\rangle \equiv \lvert ++\rangle$	$\left\lvert \frac{1}{2}\frac{1}{2}11\right\rangle \equiv \lvert 11\rangle$
$\left\lvert \frac{1}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2}\right\rangle \equiv \lvert +-\rangle$	$\left\lvert \frac{1}{2}\frac{1}{2}10\right\rangle \equiv \lvert 10\rangle$
$\left\lvert \frac{1}{2}\frac{1}{2}-\frac{1}{2}\frac{1}{2}\right\rangle \equiv \lvert -+\rangle$	$\left\lvert \frac{1}{2}\frac{1}{2}1-1\right\rangle \equiv \lvert 1-1\rangle$

$\left|\frac{1}{2}\frac{1}{2}-\frac{1}{2}-\frac{1}{2}\right\rangle \equiv |--\rangle$ $\left|\frac{1}{2}\frac{1}{2}00\right\rangle \equiv |00\rangle$

Due to (45), it is clear that the first eigenvector in the left row and the first eigenvector in the right now must be parallel. Therefore, we can choose

$$|11\rangle := |++\rangle . \tag{46}$$

Now, we climb down by means of the ladder operator

$$\hat{S}_- = \hat{S}_1 - i\hat{S}_2 \equiv \hat{S}_-^{(1)} + \hat{S}_-^{(2)}.$$

Applying $\hat{S}_-$ to (46) and using the formula

$$\hat{J}_\pm | jm\rangle = \hbar\sqrt{j(j+1)-m(m\pm 1)}| jm\pm 1\rangle , \tag{47}$$

We obtain

$$\hat{S}_-|11\rangle = \hbar\sqrt{2}|10\rangle$$

For the left hand side and

$$\hat{S}_-|++\rangle = \left(\hat{S}_-^{(1)}\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\right)\otimes\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\otimes\left(\hat{S}_-^{(2)}\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\right)$$

$$= \hbar\left|\frac{1}{2}-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2}\frac{1}{2}\right\rangle + \hbar\left|\frac{1}{2}\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2}-\frac{1}{2}\right\rangle$$

$$= \hbar|-+\rangle + \hbar|+-\rangle$$

For thr right hand side. Hence,

$$|10\rangle = \frac{1}{\sqrt{2}}|-+\rangle + \frac{1}{\sqrt{2}}|+-\rangle) \tag{48}$$

The coefficients $\frac{1}{\sqrt{2}}$ are called Clebsh-Gordan coefficients. Applying $\hat{S}_-$ once again to (48), we obtain

$$\hat{S}_-|10\rangle = \hbar\sqrt{2}|11\rangle$$

For the left hand side and

$$\frac{1}{\sqrt{2}}\hat{S}_-\left(|-+\rangle+|+-\rangle\right)=\frac{1}{\sqrt{2}}\left(\hat{S}_-^{(1)}\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle\right)\otimes\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle+\frac{1}{\sqrt{2}}\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle\otimes\left(\hat{S}_-^{(2)}\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\right)$$

$$+\frac{1}{\sqrt{2}}\left(\hat{S}_-^{(1)}\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\right)\otimes\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle+\frac{1}{\sqrt{2}}\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\otimes\left(\hat{S}_-^{(2)}\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle\right)$$

$$=0+\frac{\hbar}{\sqrt{2}}\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\otimes\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle+\frac{\hbar}{\sqrt{2}}\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle\otimes\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle+0$$

$$=\hbar\sqrt{2}|--\rangle$$

Hence,

$$|1-1\rangle=|--\rangle$$

To determine $|00\rangle$, we expand it,

$$|00\rangle=A|++\rangle+B|+-\rangle+C|-+\rangle+D|--\rangle$$

Where

$$|A|^2+|B|^2+|C|^2+|D|^2=1$$

As the eigenspaces of the self-adjoint operator $\hat{S}^2$ are mutually orthogonal, $|00\rangle$ must be orthogonal to $|11\rangle$, $|10\rangle$ and $|1-1\rangle$. This yields

$$|11|00\rangle=A=0,\qquad \langle 10|00\rangle=B+C=0,\quad \langle 1-1|00\rangle=D=0$$

In accordance with the Condon-Shortley convention Cornwell [9] we choose B to be positive. Then, normalization yields

$$|00\rangle=\frac{1}{\sqrt{2}}|+-\rangle-\frac{1}{\sqrt{2}}|-+\rangle$$

Thus, we have completed the construction of thr common eigenbasis of $\left(\hat{S}^{(1)}\right)^2,\left(\hat{S}^{(2)}\right)^2,\hat{S}^2$ and $\hat{S}^3$. We summarize:

$$|11\rangle = |++\rangle, \quad |10\rangle = \frac{1}{\sqrt{2}}|+-\rangle + \frac{1}{\sqrt{2}}|-+\rangle, \quad |1-1\rangle = |--\rangle$$

$$|00\rangle = \frac{1}{\sqrt{2}}|+-\rangle - \frac{1}{\sqrt{2}}|-+\rangle,$$

In matrix form, the change of basis reads

$$\begin{bmatrix} |11\rangle \\ |10\rangle \\ |1-1\rangle \\ |00\rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1/2} & \sqrt{1/2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \sqrt{1/2} & -\sqrt{1/2} & 0 \end{bmatrix} \begin{bmatrix} |++\rangle \\ |+-\rangle \\ |-+\rangle \\ |--\rangle \end{bmatrix}$$

Remark

By similar computations, we obtain the Clebsch-Gordan coefficients for $j_2 = \frac{1}{2}$ and arbitrary values of j_1:

$\left\langle j_1 \frac{1}{2} m_1 m_2 \middle| j_1 \frac{1}{2} sm \right\rangle$	$m_2 = \frac{1}{2}$	$m_2 = -\frac{1}{2}$
$j = j_1 + \frac{1}{2}$	$\sqrt{\frac{j_1 + m + 1/2}{2j_1 + 1}}$	$\sqrt{\frac{j_1 - m + 1/2}{2j_1 + 1}}$
$j = j_1 - \frac{1}{2}$	$-\sqrt{\frac{j_1 + m + 1/2}{2j_1 + 1}}$	$\sqrt{\frac{j_1 + m + 1/2}{2j_1 + 1}}$

Theorem 2. *the two bases* $|j_1 j_2 m_1 m_2\rangle$ *and* $|j_1 j_2 jm\rangle$ *are related by*

$$|j_1 j_2 jm\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1 j_2 m_1 m_2 | j_1 j_2 jm \rangle |j_1 j_2 m_1 m_2\rangle,$$

$$|j_1 j_2 m_1 m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} \langle j_1 j_2 jm | j_1 j_2 m_1 m_2 \rangle |j_1 j_2 jm\rangle,$$

Proof. For given values of j_1 and j_2, the values of j are restricted by the condition [10].

$$j_1 + j_2 \geq j \geq |j_1 - j_2|$$

And j runs from $j_1 + j_2$ down to $|j_1 - j_2|$ in integer steps. For $j = j_1 + j_2$, the Clebsch-Gordan coefficients

$\langle j_1 j_2 m_1 m_2 | j_1 j_2 jm \rangle$ can be read off from the sequence of equations obtained by repeated application of the ladder operator $\hat{J}_-$ to

$$| j_1, j_2, j_1 + j_2, j_1 + j_2 \rangle = | j_1 j_2 j_1 j_2 \rangle,$$

As an example, consider the first step. Using 6.5, we obtain

$$\hat{J}_- | j_1, j_2, j_1 + j_2, j_1 + j_2 \rangle = \sqrt{2(j_1 + j_2)} | j_1, j_2, j_1 + j_2, j_1 + j_2 - 1 \rangle$$

For the left hand side and

$$\hat{J}_- | j_1 j_2 j_1 j_2 \rangle = \left(\hat{J}_-^{(1)} | j_1 j_1 \rangle \right) \otimes | j_2 j_2 \rangle + | j_1 j_1 \rangle \otimes \left(\hat{J}_-^{(2)} | j_2 j_2 \rangle \right)$$

$$= \sqrt{2 j_1} | j_1 j_1 - 1 \rangle \otimes | j_2 j_2 \rangle + \sqrt{2 j_2} | j_1 j_1 \rangle \otimes | j_2 j_2 - 1 \rangle$$

For the right hand side. This yields

$$| j_1, j_2, j_1 + j_2, j_1 + j_2 - 1 \rangle = \sqrt{\frac{j_1}{j_1 + j_2}} | j_1, j_2, j_1 - 1, j_2 \rangle + \sqrt{\frac{j_2}{j_1 + j_2}} | j_1, j_2, j_1, j_2 - 1 \rangle$$

We read off

$$\langle j_1 j_2 m_1 m_2 | j_1, j_2, j_1 + j_2, j_1 + j_2 - 1 \rangle = \begin{cases} \sqrt{\frac{j_1}{j_1 + j_2}} & m_1 = j_1 - 1, m_2 = j_2 \\ \sqrt{\frac{j_2}{j_1 + j_2}} & m_1 = j_1, m_2 = j_2 - 1 \\ 0 & otherwise \end{cases}$$

(in view of 6.3, it is clear that $\langle j_1 j_2 m_1 m_2 | j_1 j_2 jm \rangle = 0$ unless $m = m_1 + m_2$).

For $j = j_1 + j_2 - 1$ etc., one first has to choose a vector $| j_1 j_2 jj \rangle$ in such a way that it is orthogonal to all vectors $| j_1 j_2 j'j \rangle, j_1 + j_2 \geq j' >$ found before. Then, application of $\hat{J}_-$ o the expansion of this vector in the basis $| j_1 j_2 m_1 m_2 \rangle$ yields a sequence of equations from which the Clebsch-Gordan coefficients $\langle j_1 j_2 m_1 m_2 | j_1 j_2 jm \rangle$ with the fixed value of *j* under consideration can be read off.

Remark

The reduction procedure applied in he proof of Theorem 2 yields another proof of Theorem 1. indeed, the wave functions $\left|s_1 s_2 m_1 m_2\right\rangle$ and $\left|s_1 s_2 sm\right\rangle$ form bases in the tensor representation space $V = V_{j_1} \otimes V_{j_2}$. This space has dimension

$$\dim(V) = (2s_1+1)(2s_2+1) = \sum_{s=|s_1-s_2}^{s_1+s_2}$$

Under group transformations, the functions $\left|s_1 s_2 m_1 m_2\right\rangle$ transform according to the representation $D^{s_1} \otimes D^{s_2}$ of SU(2):

$$\left|s_1 s_2 m_1 m_2\right\rangle \mapsto \left|s_1 s_2 m_1 m_2\right\rangle' = \left(D^{s_1} \otimes D^{s_2}\right)(a)\left|s_1 s_2 m_1 m_2\right\rangle$$

$$= \left(\sum_{n_1} D^{s_1}_{m_1 n_1}(a)\left|s_1 n_1\right\rangle\right) \otimes \left(\sum_{n_2} D^{s_2}_{m_2 n_2}(a)\left|s_2 n_2\right\rangle\right)$$

$$= \sum_{n_1, n_2} D^{s_1}_{m_1 n_1}(a) D^{s_2}_{m_2 n_2}(a)\left|s_1 s_2 m_1 m_2\right\rangle$$

Where $a \in$ SU(2). On the other hand, the wave functions $\left|s_1 s_2 sm\right\rangle$ transform according to the representation D^s:

$$\left|s_1 s_2 sm\right\rangle \mapsto \left|s_1 s_2 sm\right\rangle' = D^s(a)\left|s_1 s_2 sm\right\rangle = \sum_n D^s_{mn}\left|s_1 s_2 sn\right\rangle$$

It follows that the transformation of *V* which transforms the basis $\left|s_1 s_2 m_1 m_2\right\rangle$ into the basis $\left|s_1 s_2 sm\right\rangle$ provides the following equivalence of representations of SU(2):

$$D^{s_1} \otimes D^{s_2} = \sum_{s=|s_1-s_2|}^{s_1+s_2} D^s$$

The relations between the corresponding representations matrices are given by

$$D^s_{mn} = \left\langle s_1 s_2 sm\right|\left(D^{s_1} \otimes D^{s_2}\right)(a)\left|s_1 s_2 sn\right\rangle$$

$$= \sum_{\substack{m_1+m_2=m \\ n_1+n_2=n}} \langle s_1 s_2 sm | s_1 s_2 m_1 m_2 \rangle \langle s_1 s_2 m_1 m_2 | (D^{s_1} \otimes D^{s_2})(a) | s_1 s_2 n_1 n_2 \rangle \times \ldots\ldots\ldots \times \langle s_1 s_2 n_1 n_2 | s_1 s_2 sn \rangle$$

$$= \sum_{\substack{m_1+m_2=m \\ n_1+n_2=n}} \langle s_1 s_2 sm | s_1 s_2 m_1 m_2 \rangle D^{s_1}_{m_1 n_1}(a) D^{s_2}_{m_2 n_2}(a) \langle s_1 s_2 n_1 n_2 | s_1 s_2 sn \rangle$$

And, analoously,

$$D^{s_1}_{m_1 n_1}(a) D^{s_2}_{m_2 n}(a) = \langle s_1 s_2 m_1 m \, | (D^{s_1} \otimes D^{s_2})(a) | s_1 s_2 n_1 n_2 \rangle$$

$$= \sum_{s=|s_1-s_2|}^{s_1+s_2} \langle s_1 s_2 m_1 m_2 | D^s(a) | s_1 s_2 n_1 n_2 \rangle$$

$$= \sum_{s=|s_1-s_2|}^{s_1+s_2} \langle s_1 s_2 m_1 m_2 | s_1 s_2 s m_2 \rangle \langle s_1 s_2 s m_2 | D^s(a) | s_1 s_2 s n_2 \rangle \times \ldots\ldots\ldots \times \langle s_1 s_2 s n_2 | s_1 s_2 n_1 n_2 \rangle$$

$$= \sum_{s=|s_1-s_2|}^{s_1+s_2} \langle s_1 s_2 m_1 m_2 | s_1 s_2 sm \rangle D^s_{mn}(a) \langle s_1 s_2 sn | s_1 s_2 n_1 n_2 \rangle$$

Where $m = m_1 + m_2$ and $n = n_1 + n_2$.

Properties of the Clebsch-Gordan coefficient

1. The Clebsch-Gordan coefficients $\langle j_1 j_2 m_1 m_2 | j_1 j_2 jm \rangle$ vanish unless $m = m_1 + m_2$ and $|j_1 - j_2| \le j \le j + j_2$
2. For each fixed value of *j*, the vectors $| j_1 j_2 jm \rangle$ are determined up to a common phase. By convention, these phase factors are chosen in such a way that

$\langle j_1 j_2 m_1 m_2 j | j_1 j_2 j \rangle$ is real and positive

Then, all Clebsch-Gordan coefficients are real.

3. The Clebsch-Gordan coefficients possess the symmetry property

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 jm \rangle = (-1)^{j+j_1+j_2} \langle j_2 j_1 m_2 m_1 | j_2 j_1 jm \rangle$$

4. The Clebsch-Gordan coefficients $\langle j_1, j_2, j_1, j-1 | j_1, j_2, j, j \rangle$ are real and positive.

5. One has orthogonality condition

$$\sum \langle j_1 j_2 m_1 m_2 \mid j_1 j_2 jm \rangle \langle j_1 j_2 m_1' m_2' \mid j_1 j_2 jm \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

4. Conclutions

From the results of the study above, it can be concluded that angular momentum in quantum mechanics has been studied, especially the appearance in the operator form, and the sum of angular momentum, eigenform values of angular momentum. Angular momentum in quantum mechanics and Clebsh-Gordan coefficient analysis.

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