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Abundant Exact Soliton Solutions to the Space-Time Fractional Phi-Four Effective Model for Quantum Effects Through the Modern Scheme

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Abstract

The space-time fractional Phi-four (PF) equation is measured as a particular case of the familiar Klein-Fock-Gordon (KFG) model and plentiful quantum effects can be investigated through the PF model's solutions. In this article, the auxiliary equation method (AEM) is employed to attain the traveling wave solutions and in this purpose, the complex wave transformation and Maple software are utilized. The constructed wave solutions are the form likely, hyperbolic, exponential, rational, and trigonometric functions as well as their integration. The physical significance of the obtained solutions for the specific values of the integrated parameters in the course of representing graphs and understood the physical phenomena. It is shown that the AEM is powerful, effective and simple and provide more general traveling wave solutions to the NLEEs.

Keywords**:** The Phi-4 model; the auxiliary equation method; nonlinear evolution equation; soliton.

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1. Introduction

In recent times, nonlinear fractional partial differential equations (NLFPDE) which is initially define in 1695, are considered as one of the significant and fundamental branches in physical science [1], particularly for disclosing various novel properties of compound phenomena in different fields such as nuclear physics, atomic physics, quantum mechanics, solid-state physics, optical physics and more. Throughout this context, several effective nonlinear models have been derived by scholars to investigate the exact solution of FPDEs in multiple studies such as the first integral method [2], the auxiliary equation method [3-6], the two variable $(G'/G\,,1/G)$-expansion method [7], the extended tanh-function method [8], the transformed rational function method [9], the variational iteration method [10], the finite difference method [11], the Sine-Gordon expansion Method [12], the MSE method [13-15], the modified Kudryashov method[16],the Fourier transform method [17], the Exp-function method [18], the Modified $exp^{(-\Omega(\xi))}$-expansion function method [19], the test function method [20], the (G'/G)-expansion method [21-24], the Logarithmic transformation method [25], the fractional sub-equation method [26], etc. In this study, we consider the well-known nonlinear space-time fractional Phi-four (PF) equation [37] such as:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}-\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}+u(x,t)-u^3(x,t)=0;\, t>0,\quad and\;\; 0<\alpha\leq 1. \tag{1.1}$$

Here$\alpha$is the order of fractional derivative. In recent years, the nonlinear space-time fractional Phi-four equation is handled through a variety of well-organized and powerful methods for developing several solitary traveling wave solutions. For example, the approach ofexp$(-\Phi(\xi)$ and the modified Kudryashov[27], the generalized Kudryashov method [28],the New extended direct algebraic scheme[29], the Weierstrass elliptic function method[30], the Mapping technique[31], the unified technique [32], the modified Simple equation method[33], tanh function method[34], etc. are the leading approaches to explore the exact solutions of the Phi-four model. Here, our declared approach is efficient and deliver more general closed form traveling wave solutions to the fractional NPDEs. As per statistical assessment, there are no fruitful studies found yet on our preferred Phi-four equation by the aid of our declared auxiliary equation method to extract traveling wave solutions with the conformable derivative. The main objective of this study is to exact more general close form wave solutions of space- time fractional Phi-four equation by applying the auxiliary equation method. We have compare our result with the result found from the generalized (G'/G)-expansion method. We also have depicted the attained solutions by taking the suitable values of parameters and explain the physical important. So, we can claim that our proposed study on space-time fractional Phi-four equation with the aid of the auxiliary equation method is novel in the sense of conformable derivative. The rest of the article is set in the following: We sketch the definition of the conformable derivative in section 2. In section 3, we explain the method briefly. In section 4, we extract the traveling wave solutions and compare the results of the attained solitons and finally we discuss the conclusions.

2. The Conformable Fractional Derivative (CFD)

The conformable fractional derivative (CFD) of order $\iota>0$ with respect variable $t$is recognized by Khalil and his colleagues [35] is defined as follows:

Consider a function$g: (0, \infty) \to \mathbb{R}$, the conformable fractional derivative of g is define as.

$$\mathcal{L}_t^{\iota} g(t) = \lim_{\varepsilon \to 0} \frac{g\left(t+\varepsilon t^{1-\iota}\right)-g(t)}{\varepsilon} \, \forall t > 0 \text{ and } \iota \epsilon (0,1].$$

The following theorems pass on to the properties fulfilled by the definition:

Theorem 1.Let us consider$\iota \epsilon (0,1]$ and g, Q are ι-differentiable at a point$t$. Then the following properties carry:

- $\mathcal{L}_t^{\iota}(cg + dQ) = c\mathcal{L}_t^{\iota}(g) + d\mathcal{L}_t^{\iota}(Q)$
- $\mathcal{L}_t^{\iota}(t^q) = qt^{q-\iota}, \ \forall q \in \mathbb{R}$
- $\mathcal{L}_t^{\iota}(\tau) = 0, \ \forall u(t) = \tau.$
- $\mathcal{L}_t^{\iota}(Qg) = Q\mathcal{L}_t^{\iota}(g) + g\mathcal{L}_t^{\iota}(Q)$
- $\mathcal{L}_t^{\iota}\left(\frac{Q}{g}\right) = \frac{g\mathcal{L}_t^{\iota}(Q) - Q\mathcal{L}_t^{\iota}(g)}{g^2}$
- $\mathcal{L}_t^{\iota} Q(t) = t^{1-\iota} \frac{dQ}{dt}$ wherein $t^{1-\iota}$ indicate a fractional conformable function

for all $c, d \in \mathbb{R}$ [35].

Conformable differential operator satisfies some important types of stuff like the chain law, Taylor series expansion and Laplace transforms [36].

Theorem 2.Let$Q = Q(t)$be a ι conformable differentiable function and g is differentiable in the range of Q.Then $\mathcal{L}_t^{\iota}(Qog)(t) = t^{1-\iota} g'(t) Q'\left(g(t)\right)$.

3. Method Description

Let a general FNLDE is as follows:

$$\mathcal{D}\left(\upsilon, D_t^{\alpha}\upsilon, D_x^{\beta}\upsilon, D_y^{\gamma}\upsilon, D_t^{2\alpha}\upsilon \ldots\right) = 0, \tag{3.1}$$

where $\upsilon = \upsilon(t, x, y, z)$ is wave function, $\mathcal{D}$ is a polynomial in$\upsilon(t, x, y, z)$ and its partial derivatives. It contains the utmost order nonlinear terms and derivatives, the subscripts specify partial derivatives. To find the solution of (3.1) using the auxiliary equation method, it holds the following steps:

Step 1: Consider the traveling wave variable

$$\upsilon(x, y, t) = \upsilon(\varphi); \varphi = m\frac{x^{\kappa}}{\kappa} + n\frac{y^{\mu}}{\mu} \pm k\frac{t^{\rho}}{\rho}, \tag{3.2}$$

where k is the traveling wave velocity and $\rho, \kappa, \mu$are fractional order derivatives $(0 < \rho \leq 1, \kappa, \mu)$ and they may be equal or not. Applying the transformation (3.2) into (3.1) which allows us to transform the (3.1) into the ordinary differential equation (ODE) as:

$$\mathcal{Z}(\upsilon, \upsilon', \upsilon'', \dots) = 0, \tag{3.3}$$

where the polynomial $\mathcal{Z}$ contain $\upsilon(\varphi)$ and its different derivatives, wherein $\upsilon'(\varphi) = \frac{d\upsilon}{d\varphi}$.

Step 2: Here (3.3) can be integrated term by term one or more times.

Step 3: Suppose that the traveling wave solution of (3.3) can be revealed in the form:

$$\upsilon(\varphi) = \sum_{i=0}^{N} c_i\, a^{if(\varphi)}, \tag{3.4}$$

Where the constants c_i and a have to be calculated, Thus $c_N \neq 0$ and $f(\varphi)$ satisfies the subsequent auxiliary equation:

$$f'(\varphi) = \frac{1}{lna}\left\{pa^{-f(\varphi)} + q + ra^{f(\varphi)}\right\}. \tag{3.5}$$

The prime locates for derivative with respect to φ and p, q, r are real parameters.

Step 4: The positive integer N occurs in (3.4) can be examined by the homogeneous balancing the derivatives of highest order and the highest order nonlinear terms arise in (3.3).

Step 5: Putting (3.4) together with (3.5) into (3.3) and the value of N determined in Step 4, we attain polynomial in $a^{if(\varphi)}$. Taking all the terms of the same power of $a^{if(\varphi)}$, where $(i = 1, 2, 3 \dots)$ and equating them to zero yields a system of algebraic equations with the values c_i, p, q and r and solving the algebraic equations give the values of the unknown parameters. As the general solution of (3.5) is known, putting the values of $c_i (i = 1, 2, 3 \dots), p, q$ and r into (3.5), we find out more general types and fresh closed form soliton of the fractional NLDE (3.3).

Step 6: For the different values of p, q and r and their relationship, (3.5) provides different types of general solutions.

4. Formulation of the Solutions and Results Discussion

In this section, we study the space-time fractional Phi-four (PF) equation to estimate several newer and further general closed form wave solutions by applying the auxiliary equation method and compare our results with the results found from the generalized (G'/G)-expansion method. Moreover, we have discussed on the graphical depiction and physical clarification for the different natures of attained soliton solutions.

4.1. Formulation of the solutions to the space-time fractional Phi-four (PF)model

In this section, we prepare several advance and closed form soliton solutions to the nonlinear space-time fractional Phi-four equation. Now we study the well-known nonlinear conformable space-time fractional Phi-four (PF) model [37] in Eq. (1.1).

Consider the travelling wave variable $v(\varphi) = v(x,t)$, where $\varphi = m\frac{x^\alpha}{\alpha} + k\frac{t^\alpha}{\alpha}$then Eq.(1.1) is reduced to an ODE as:

$$(k^2 - m^2)v'' + v - v^3 = 0 \tag{4.1}$$

The solutions archived by applying the auxiliary equation method to space-time fractional Phi-four equation are given below. Balancing between the uppermost order linear and nonlinear terms appearing in (4.1), gives us $N = 1$. Therefore, the solution of (4.1) is:

$$v = c_0 + c_1 a^{f(\varphi)}. \tag{4.2}$$

By applying the results of (4.2), (3.5) and (4.1) and connecting the coefficients the powers of $a^{if(\varphi)}$ and taking zero, we get a set of algebraic equations (for simplicity which are not assembled here) for c_0 , c_1, p, q, r. Solving the system of algebraic equations by means of the characteristic computation software Maple, recommend the solutions as:

$$c_0 = \pm\frac{q}{\sqrt{q^2-4pr}}, \quad c_1 = \pm\frac{2r}{\sqrt{q^2-4pr}}, \quad k = \pm\sqrt{m^2 + \frac{2}{q^2-4pr}}. \tag{4.3}$$

Family 1: When $q^2 - 4\,pr < 0$ and $r \neq 0$, we reach the soliton solutions:

$$v_1(x,t) = \pm i\tan\left(\frac{\sqrt{4\,pr-q^2}}{2}\varphi\right), \tag{4.4}$$

$$v_2(x,t) = \mp i\cot\left(\frac{\sqrt{4\,pr-q^2}}{2}\varphi\right). \tag{4.5}$$

Family 2: When $q^2 - 4\,pr > 0$ and $r \neq 0$, the solutions turn into the form:

$$v_3(x,t) = \mp\tanh\left(\frac{\sqrt{q^2-4\,pr}}{2}\varphi\right), \tag{4.6}$$

$$v_4(x,t) = \mp\coth\left(\frac{\sqrt{q^2-4\,pr}}{2}\varphi\right). \tag{4.7}$$

Family 3: When $q^2 + 4\,p^2 < 0$, $r \neq 0$ and $r = -p$, we achieved the solutions:

$$v_5(x,t) = \pm i\tan\left(\frac{\sqrt{-(q^2+4\,p^2)}}{2}\varphi\right), \tag{4.8}$$

$$v_6(x,t) = \mp i\cot\left(\frac{\sqrt{-(q^2+4\,p^2)}}{2}\varphi\right). \tag{4.9}$$

Family 4: When $q^2 + 4\,p^2 > 0$, $r \neq 0$ and $r = -p$, we get the solutions:

$$v_7(x,t) = \mp\tanh\left(\frac{\sqrt{(q^2+4\,p^2)}}{2}\varphi\right),$$
(4.10)

$$v_8(x,t) = \mp\coth\left(\frac{\sqrt{(q^2+4\,p^2)}}{2}\varphi\right).$$
(4.11)

Family 5: When $q^2 - 4\,p^2 < 0$ and $r = p$, the solutions turn into the form:

$$v_9(x,t) = \pm i\tan\left(\frac{\sqrt{-(q^2-4\,p^2)}}{2}\varphi\right),$$
(4.12)

$$v_{10}(x,t) = \mp i\cot\left(\frac{\sqrt{-(q^2-4\,p^2)}}{2}\varphi\right).$$
(4.13)

Family 6: When $q^2 - 4\,p^2 > 0$ and $r = p$, our attained the solutions:

$$v_{11}(x,t) = \mp\tanh\left(\frac{\sqrt{(q^2-4\,p^2)}}{2}\varphi\right),$$
(4.14)

$$v_{12}(x,t) = \mp\coth\left(\frac{\sqrt{(q^2-4\,p^2)}}{2}\varphi\right).$$
(4.15)

Family 7: When $q^2 = 4pr$, we found the trivial solution. Therefore, the solution omitted here.

Family 8: When $rp < 0$, $q = 0$ and $r \neq 0$, we found:

$$v_{13}(x,t) = \mp\tanh\left(\sqrt{-rp}\varphi\right),$$
(4.16)

$$v_{14}(x,t) = \mp\coth\left(\sqrt{-rp}\varphi\right).$$
(4.17)

Family 9: When $q = 0$ and $p = -r$, we achieved the solution:

$$v_{15}(x,t) = \pm\left(\frac{1+e^{(-2\,r\varphi)}}{-1+e^{(-2\,r\varphi)}}\right).$$
(4.18)

Family 10: When $p = r = 0$, the obtained solution has no physical significance and not written here.

Family 11: When $p = q = h$ and $r = 0$, we attain the soliton solution in the form of trivial solution and the solution has no physical significance. Therefore, the solution omitted here.

Family 12: When $q = r = h$ and $p = 0$, we get the solution as:

$$v_{16}(x,t) = \pm\left(\frac{2e^{h\varphi}}{1-e^{h\varphi}}\right). \tag{4.19}$$

Family 13: When $q = p + r$, we have:

$$v_{17}(x,t) = \pm\left(\frac{p+r}{p-r} - \frac{2r}{p-r}\left(\frac{1-pe^{(p-r)\,\varphi}}{1-re^{(p-r)\,\varphi}}\right)\right). \tag{4.20}$$

Family 14: When $q = -(p + r)$, the obtained solution:

$$v_{18}(x,t) = \pm\left(\frac{-p-r}{p-r} + \frac{2r}{p-r}\left(\frac{p-e^{(p-r)\,\varphi}}{r-e^{(p-r)\varphi}}\right)\right). \tag{4.21}$$

Family 15: When $p = 0$, we get the solution:

$$v_{19}(x,t) = \pm\left(\frac{2re^{q\varphi}}{1-re^{q\varphi}}\right). \tag{4.22}$$

Family 16: When $r = q = p \neq 0$, we obtain:

$$v_{20}(x,t) = \mp i\tan\left(\frac{\sqrt{3}}{2}p\varphi\right). \tag{4.23}$$

Family 17: When $r = q = 0$, the soliton solution turns into trivial form which is not shown here.

Family 18: When $p = q = 0$, the obtained solution has no physical significance and not written here.

Family 19: When $r = p$ and $q = 0$, the establish solutions:

$$v_{21}(x,t) = \mp i\tan(p\varphi). \tag{4.24}$$

Family 20: When $r = 0$, we found the trivial solution. Therefore, the solution omitted here.

In the above soliton solutions, $\varphi = m\frac{x^\alpha}{\alpha} + k\frac{t^\alpha}{\alpha}$ and k is the wave velocity and p, q, r, m are arbitrary constant.

4.2. Graphical depiction and physical significance of the attained solutions

In this section, the Wolfram Mathematica is used to depict the attained solutions for the distinct values of the involved parameters and discuss the physical significance. For simplicity, some figures of the obtained solutions are exposed here and some are skipped which scheduled below:

We discuss on the graphical depiction and physical explanation for the various natures of obtained soliton solutions such as, (4.4), (4.10), (4.13), (4.19) and (4.24) of the considered model in this work for the diverse values of all incorporated parameters in suitable interval. But, for minimalism the figures of the remaining other obtained solutions are omitted here. The effect of the values of fractional parameter discussed below:

The outlines of the solution (4.4) for different values of fractional parameter $\alpha$are:

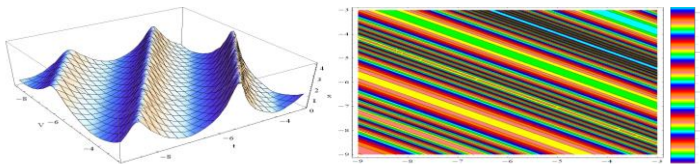

Figure 1: 3D and contour plot of (4.4) which is the periodic soliton for $p = 1.5, q = 0.5, r = 1.5, m = 0.7, \alpha = 0.99$.

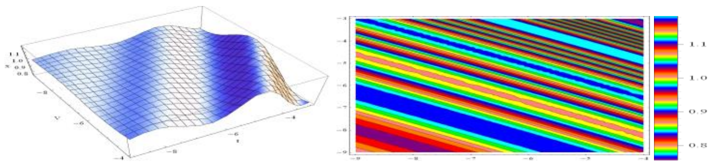

Figure 2: 3D and contour plot of (4.4) which is the periodic soliton for $p = r = 1.5$, $q = 0.5$, $m = 0.7$, $\alpha = 0.95$

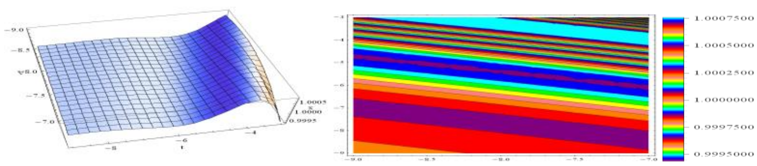

Figure 3: 3D and contour plot of (4.4) which is periodic soliton for $p = 1.5, q = 0.5, r = 1.5, m = 0.7, \alpha = 0.85$.

The above figures (Fig.1-Fig.3) for the solution (4.4) it is seen from the analysis that, the shape of the soliton (4.4) is multiple periodic (Fig.1-Fig.3) for the value of fractional parameter $\alpha = 0.99$, $\alpha = 0.95$ and $\alpha = 0.80$. But, when the value of α varies with respect to time, the shape of these 3D figures are changed (see in Fig.1-Fig.3).

And the profiles of the solution (4.10) for diverse values of fractional parameter α are:

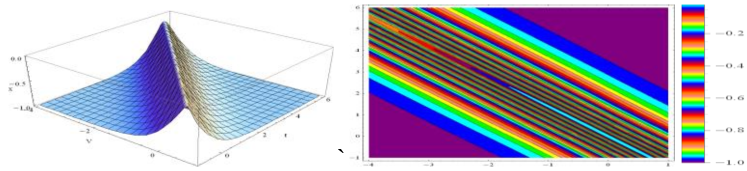

Figure 4: Plot of 3D and contour shape of (4.10) which is squeezed bell shape soliton for $p = 1.5, q = 0.5, r = 1.5,\ m = 0.7, \alpha = 0.99$.

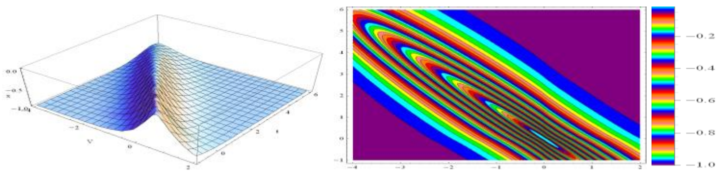

Figure 5: Design of 3D and contour plot of (4.10) which is a general soliton for$p = 1.5, q = 0.5, r = 1.5,\ m = 0.7, \alpha = 0.95$.

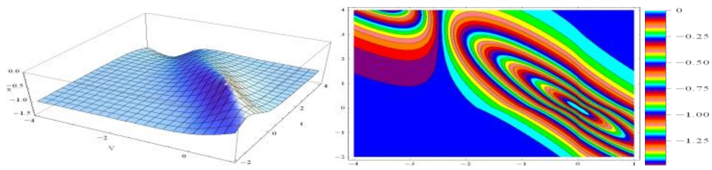

Figure 6: Plot of 3D design and contour shape of (4.10) which is a general soliton for $p = 1.5, q = 0.5, r = 1.5$, $m = 0.7, \alpha = 0.85$.

Again, the figures (Fig.4-Fig.6) for the solution (4.10) it is seen from the investigation that, the shape of the soliton (4.10) is kink shape soliton (Fig.4-Fig.6) for the value of fractional parameter $\alpha = 0.99$, $\alpha = 0.95$and$\alpha = 0.85$. But, when the value of α varies with respect to time, the outline of these 3D figures are changed (see in Fig.4-Fig.6).

The outlines of the wave solution (4.13) for different values of fractional parameter α are:

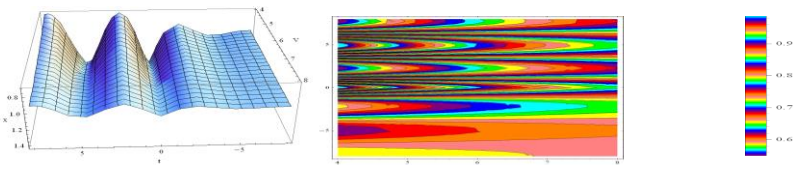

Figure 7: Design of 3D and contour plot of (4.13) which is periodic soliton for $p = -0.5, q = 3.5, r = 0.5, m = 0.4, \alpha = 0.99$.

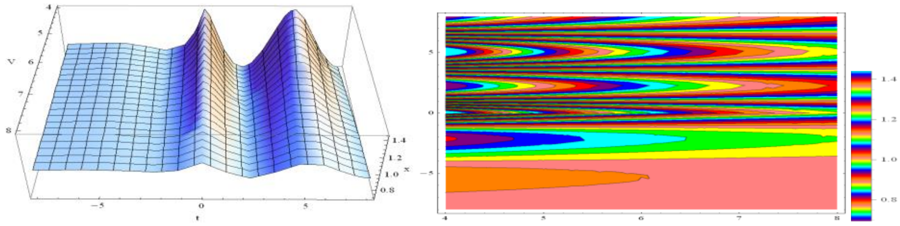

Figure 8: Plot of 3D and contour shape of (4.13) which is periodic soliton for$p = -0.5, q = 3.5, r = 0.5, m = 0.4, \alpha = 0.95$.

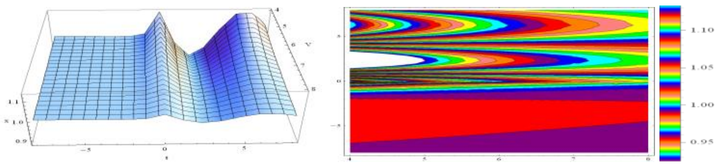

Figure 9: Plot of 3D and contour shape of (4.13) which is periodic soliton for $p = -0.5, q = 3.5, r = 0.5, m = 0.4, \alpha = 0.90$.

Furthermore, in the above graphical analysis, we draw the 3D plot as, Fig.7, Fig.8, Fig.9 and their contour shapes for the value of the fractional parameter $\alpha = 0.99, \alpha = 0.95$ and $\alpha = 0.90$ respectively of soliton (4.13). In this analysis, we examine that, the nature of figures are changed (see in Fig.7-Fig.9) when the values of fractional parameter changed.

The outlines of the wave solution (4.19) for diverse values of fractional parameter α are:

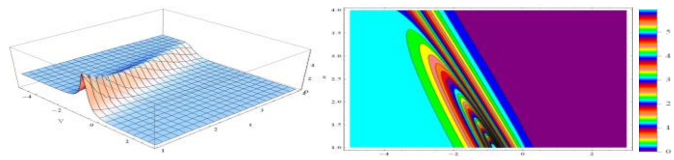

Figure 10: Plot of 3D design and contour shape of (4.19) which is brether type soliton for $p = 0.0, q = r = h = -1, m = 2, \alpha = .95$.

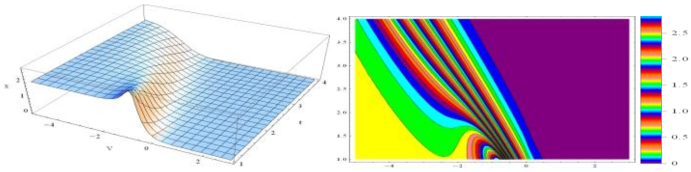

Figure 11: Design of 3D shape and contour plot of (4.19) which kink shape soliton for $p = 0.0, q = r = h = -1, m = 2, \alpha = .90$

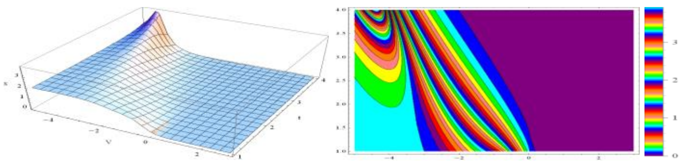

Figure 12: Plot of 3D design and contour shape of (4.19) which is a breather type soliton for $p = 0.0, q = r = h = -1, m = 2, \alpha = .80$.

From the above figures (Fig.10-Fig.12) for the solution (4.19) we analysis that, the shape of the soliton (4.19) is kink shape soliton (see Fig.10-Fig.12) for the value of fractional parameter $\alpha = 0.95$, $\alpha = 0.90$ and $\alpha = 0.80$ and the contour shape of this soliton for the similar value is depicted at $t = 0$ (seeFig.10-Fig.12). But, when the value of αvaries with respect to time, the shape of these 3D figures are changed (see Fig.10-Fig.12).

The outlines of the wave solution (4.24) for different values of fractional parameter α are:

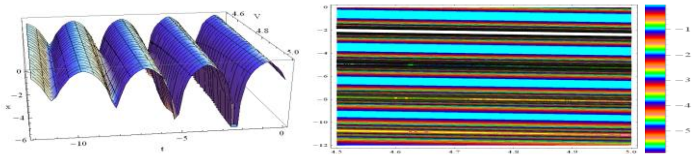

Figure 13: Design of 3D shape and contour plot of (4.24) which is a periodic soliton for $p = r = 1.1, m = 1, \alpha = .99$.

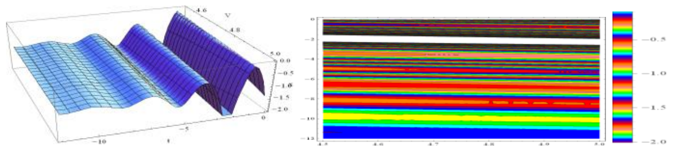

Figure 14: 3D and contour shape of (4.24) which is a periodic soliton for $q = 0,\ p = r = 1.1,\ m = 1, \alpha = .95$.

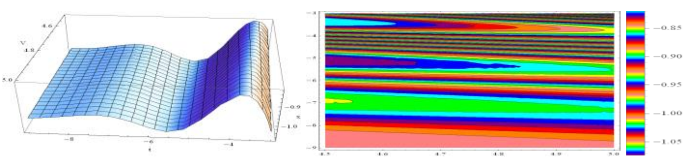

Figure 15: Design of 3D and contour shape of (4.24) which is a general soliton for $q = 0,\ p = r = 1.1, m = 1, \alpha = .9$.

From the above graphical analysis, we draw the 3D plot as, Fig.13, Fig.14, Fig.15 and their contour shapes for the value of the fractional parameter$\alpha = 0.99, \alpha = 0.95$ and $\alpha = 0.90$ respectively of soliton (4.24). Here, we examine that, the nature of figures are changed (see in Fig.13-Fig.15) when the values of fractional parameter changed.

The attained solutions of considered model are might be supportive to exemplify the internal mechanisms of the corporeal phenomena associated with the considered model. All of our obtained solutions involve various traveling wave solutions which might be disclose various novel properties of complex incidents in different fields such as nuclear physics, plasma physics solid-state physics, optical physics, atomic physics and more. The solutions are derived involving to trigonometric, hyperbolic, rational and exponential functions. We also observe that, the solutions carry different nature of familiar shapes of soliton.

4.2. Comparison of the attained solutions

In this section, we have compared the exact travelling wave solutions of the space-time fractional Phi-four model obtained through the auxiliary equation method with those solutions obtained by the generalized (G'/G)-expansion method. It is noteworthy to observe that the obtained solutions are suitable, efficient and further general. The established solutions might be useful to analyze the physical significance for the mentioned equations.

Roy *and his colleagues* [37] examined the space-time fractional Phi-four model and obtained only nine solutions (see Appendix A) by applying the generalized (G'/G)-expansion method. We detect that some of our obtained solutions are identical to the Roy *and his colleagues* solutions and some are different. In Table 1, we compare the solutions examined by the two methods:

Table 1: comparison of the solutions examined by the two methods**.**

Solutions obtained by the generalized (G'/G)-expansion method	**Solutions obtained by the auxiliary equation method**
1. $u_1 = \pm\sqrt{\frac{\rho}{Q^2+4S\psi}}\,tanh\left(\frac{\sqrt{\rho}}{2\psi}\xi\right)$	1. If $\rho = -Q^2 - 4S\psi$, $q^2 - 4\,pr = \frac{\rho}{\psi^2}, \xi = \varphi$, then $v_3 = \pm\sqrt{\frac{\rho}{Q^2+4S\psi}}\,tanh\left(\frac{\sqrt{\rho}}{2\psi}\xi\right)$
2. $u_2 = \pm\sqrt{\frac{\rho}{Q^2+4S\psi}}\,coth\left(\frac{\sqrt{\rho}}{2\psi}\xi\right)$	2. If $\rho = -Q^2 - 4S\psi$, $q^2 - 4\,pr = \frac{\rho}{\psi^2}, \xi = \varphi$, then $v_4 = \pm\sqrt{\frac{\rho}{Q^2+4S\psi}}\,coth\left(\frac{\sqrt{\rho}}{2\psi}\xi\right)$
3. $u_3 = \mp i\sqrt{\frac{\rho}{Q^2+4S\psi}}\,tan\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right)$	3. If $\rho = -Q^2 - 4S\psi, q^2 - 4\,pr = \frac{\rho}{\psi^2}$, $\xi = \varphi$,then $v_1 = \mp i\sqrt{\frac{\rho}{Q^2+4S\psi}}\,tan\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right)$
4. $u_4 = \pm i\sqrt{\frac{\rho}{Q^2+4S\psi}}\,cot\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right)$	4. If $\rho = -Q^2 - 4S\psi, q^2 - 4\,pr = \frac{\rho}{\psi^2}$, $\xi = \varphi$,then $v_2 = \mp i\sqrt{\frac{\rho}{Q^2+4S\psi}}\,cot\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right)$
5. $u_6 = \pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-Q + 2\sqrt{\Delta}tanh\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)\right\}$	5. If $Q = 0, \sqrt{\Delta} = -\frac{1}{2}$, $\psi^2 = q^2 - 4\,p^2, \xi = \varphi$, then $v_{11} = \pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-Q + 2\sqrt{\Delta}\,tanh\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)\right\}$
6. $u_7 = \pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-Q + 2\sqrt{\Delta}coth\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)\right\}$	6. If $Q = 0, \sqrt{\Delta} = -\frac{1}{2}$, $\psi^2 = q^2 - 4\,p^2, \xi = \varphi$, then $v_{12} = \pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-Q + 2\sqrt{\Delta}\,coth\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)\right\}$
7. $u_8 = \pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-Q - 2i\sqrt{\Delta}tan\left(\frac{\sqrt{-\Delta}}{\psi}\xi\right)\right\}$	7. If $Q = 0, \sqrt{\Delta} = -\frac{1}{2}$, $\psi^2 = q^2 + 4\,p^2, \xi = \varphi$, then $v_5 = \pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-Q - 2i\sqrt{\Delta}\,tan\left(\frac{\sqrt{-\Delta}}{\psi}\xi\right)\right\}$
8. $u_9 = \pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-Q + 2i\sqrt{\Delta}cot\left(\frac{\sqrt{-\Delta}}{\psi}\xi\right)\right\}$	8. If $Q = 0, \sqrt{\Delta} = -\frac{1}{2}$, $\psi^2 = q^2 + 4\,p^2, \xi = \varphi$, then $v_6 = \pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-Q - 2i\sqrt{\Delta}\,cot\left(\frac{\sqrt{-\Delta}}{\psi}\xi\right)\right\}$

By applying auxiliary equation method, we have determined twenty one solutions. In the above table, we have noticed that the solutions $u_1 - u_4$ and $u_6 - u_9$ obtained by the generalized (G'/G)-expansion method and the solutions$v_1 - v_6, v_{11}$andv_{12}obtained by auxiliary equationmethod are identical. Also, the obtained solutions $v_7 - v_{10}, v_{13}, v_{14}, v_{20}, v_{21}$ by the auxiliary equation method are identical with the solutions $u_1 - u_4$ and $u_6 - u_9$ obtained by the generalized (G'/G)-expansion method (for simplicity, these results are not shown in the table). The only one solution $u_5$obtained from the generalized (G'/G)-expansion method is not identical with our established solutions. In addition of the above table, we have obtained some other valuable solutions$v_{15}, v_{16}, v_{17}, v_{18}$ andv_{19} which are not found in the Roy *and his colleagues* solutions.

Therefore, comparing between the obtained solutions and the solutions obtained by Roy *and his colleagues* solutions, we might conclude that our attained solutions are practically and further general and give many solutions than Roy *and his colleagues* obtained solutions.

5. Conclusion

In this article, the enhanced auxiliary equation method (AEM) has successfully utilized and examined broad-ranging, distinctive, fresh and more general traveling wave solutions to the important space-time fractional Phi-four (PF) model. The attained further general and new stable wave solutions which are not reported in the previous literature and these solutions are ascertained as the combination of exponential functions, hyperbolic functions and trigonometric function associated with several free parameters and provides better solutions than other method likely, the generalized (G'/G)-expansion method. The significance of the solutions established with free parameters can be important to explain the tangible phenomena. The devised algorithm is effective and can be used to unravel other nonlinear models in mathematical physics and engineering. The employed method could be implemented to other kinds of fractional differential systems to investigation the range of stability and applicability and this is the apprehension of more research.

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References

[1]. M. Dalir and M. Bashour, Applications of fractional calculus, Appl. Math. Sci., 4(21), 1021-1032, 2010.

[2]. H. Rezazadeh, J. Manafian, F. S. Khodadad and F. Nazari, Traveling wave solutions for density-dependent conformable fractional diffusion-reaction equation by the first integral method and the improved $\tan(\varphi(\xi)/2)$-expansion, Opt. Quan. Elec., 51, 121, 2018.

[3]. M. A. Akbar, N. H. M. Ali and T. Tanjim, Outset of multiple soliton solutions to the nonlinear Schrodinger equation and the coupled Burgers equation, J. Phys. Commu., 3(9), 095013, 2019.

[4]. H. Rezazadeh, A. Korkmaz, M. Eslami and S. M. M. Alizamini, A large family of optical solutions to Kundu-Eckhaus model by a new auxiliary equation method, Opt. Quan. Elec., 51(84), 2019.

[5]. M. Al-Amin, M. N. Islam and M. A. Akbar, Adequate wide-ranging closed-form wave solutions to a nonlinear biological model, Par. Diff. Equ. App. Math., 2021(4), 100042, 2021.

[6]. A. Akbulut, M. Kaplan and A. Bekir, Auxiliary equation method for fractional differential equations with modified Riemann-Liouville derivative, Int. J. Nonlin. Sci. Numer. Simula., 17(7-8), 413-420, 2016.

[7]. D. Kumar and G. C. Paul, Solitary and periodic wave solutions to the family of nonlinear conformable fractional Boussinesq-like equations, Math. Meth. Appl. Sci., 44(4), 3138-3158, 2021.

[8]. E. H. M. Zahran and M. M. A. Khater, Modified extended tanh-function method and its applications to the Bogoyavlenskii equation, Appl. Math. Model, 40(3), 1769-1775, 2017.

[9]. W. X. Ma and J. H. Lee,A transformed rational function method and exact solutions to the (3+1) dimensional Jimbo-Miwa equation, Chaos. Solit. Fract., 42(3), 1356-1363, 2009.

[10]. A. Neamaty, B. Agheli and R. Darzi, Variational iteration method and He's polynomials for time fractional partial differential equations, Prog. Frac. Diff. App., l(1), 47-55, 2015.

[11]. Y. Liu, J. Roberts and Y. Yan, A note on finite difference methods for nonlinear fractional differential equations with non-uniform meshes, Int. J. Comput., 95(6-7), 1151-1169, 2017.

[12]. H. M. Baskonus, H. Bulut and T. A. Sulaiman, New Complex Hyperbolic Structures to the Lonngren-Wave Equation by Using Sine-Gordon Expansion Method, Appl. Math. Non-lin. Sci., 4(1), 129-138, 2019**.**

[13]. M. N. Islam and M. A. Akbar, Closed form exact solutions to the higher dimensional fractional Schrodinger equation via the modified simple equation method, J. Appl. Math. Phys., 6, 90-102, 2018.

[14]. M. N. Islam, M. Asaduzzaman and M.S. Ali, Exact wave solutions to the simplified modified Camassa-Holm equation in mathematical physics. AIMS Math., 5(1), 26-41, 2019.

[15]. O. A. Ilhan, M. N. Islam and M.A. Akbar, Construction of functional closed form wave solutions to the ZKBBM equation and the Schrodinger equation, Iranian J. Sci. Tech. Trans. Mech. Eng., 45, 827-840, 2021.

[16]. K. Hosseini and R. Ansari, New exact solutions of nonlinear conformable time-fractional Boussinesq-type equation using the modified Kudryashov method, J. Wav. Ran. Compl. Med., 22(4), 628-636, 2017.

[17]. W. X. Ma and L. Zhang, Lump solutions with higher-order rational dispersion relations, Pram. J. Phys., 94(43), 2020.

[18]. M. M. El-Borai, W. G. El-Sayed and R. M. Al-Masroub, Exact solutions for time fractional coupled Whitham-Broer-Kaup equations via exp-function method, Int. Res. J. Eng. Tech., 2(6), 307-315, 2015.

[19]. H. M. Baskonus, H. Bulut and A.Atangana, Onthe complex and hyperbolic structures of the longitudinal wave equation in a magneto-electro-elastic circular rod, Smart Mater. Struct., 25(3), 035022, 2016.

[20]. S. J. Chen, X. Lü and X. F. Tang, Novel evolutionary behaviors of the mixed solutions to a generalized Burgers equation with variable coefficients, Commun. Nonlin. Sci. Num. Simul., 95, 105628, 2021.

[21]. M. N. Islam and M. A. Akbar, New exact wave solutions to the space-time fractional coupled Burger equations and the space-time fractional foam drainage equation, Cogent Phys., 5, 1422957, 18, 2018.

[22]. M. N. Islam and M. A. Akbar, Closed form solutions to the coupled space-time fractional evolution equations in mathematical physics through analytical method, J. Mech. Cont. Math. Sci., 13(2), 1-23,

2018.

[23]. M. A. Akbar and N. H. M. Ali, The alternative (G'/G)-expansion method and its applications to nonlinear partial differential equations, Int. J. Phys. Sci., 6(35), 7910-7920, 2014.

[24]. M. N. Islam and M. A. Akbar, Closed form wave solutions to the time fractional Boussinesq-type and the time fractional Zakharov-Kuznetsov equations, J. Nati. Sci. Found. Sri Lanka, 47(2), 149-160, 2019.

[25]. W. X. Ma, Y. Zhang and Y. Tang, Symbolic computation of lump solutions to a combined equation involving three types of nonlinear terms, East Asian J. Appl. Math., 10(4), 732-745, 2020.

[26]. J. F. Alzaidy, The fractional sub-equation method and exact analytical solutions for some nonlinear fractional PDEs, British J. Math. Comput. Sci., 3, 153-163, 2013.

[27]. G. Akram, F. Batool and A. Riaz, Two reliable techniques for the analytical study of conformable time-fractional Phi-4 equation, Opt. Quant. Electron., 50, 22, 2018.

[28]. F. Mahmud, M. Samsuzzoha and M. A. Akbar, The generalized Kudryashov method to obtain exact traveling wave solutions of the PHI-four equation and the Fisher equation, Results Phys., 7, 4296-4302, 2017.

[29]. H. Rezazadeh, H. Tariq, M. Eslami, M. Mirzazadeh and Q. Zhou, New exact solutions of nonlinear conformable time-fractional Phi-4 equation, Chinese J. Phys., 56(6), 2805-2816, 2018.

[30]. X. Deng, M. Zhao and X. Li, Travelling wave solutions for a nonlinear variant of the PHI-four equation, Math. Comput. Model., 49(3-4), 617-622, 2009.

[31]. Z. Körpinar, Some analytical solutions by mapping methods for nonlinear conformable time-fractional Phi-4 equation, Therm. Sci., 341, 2019.

[32]. M. A. E. Abdelrahman and H. A. Alkhidhr, Closed-form solutions to the conformable space-time fractional simplified MCH equation and time fractional Phi-4 equation, Results Phys., 18, 103294, 2020.

[33]. M. Kaplan and A. Bekir, The modified simple equation method for solving some fractional-order nonlinear equations, Pramana-J. phys., 87(1), 15, 2016.

[34]. H. Tariq and G. Akram, New approach for exact solutions of time fractional Cahn-Allen equation and time fractional Phi-4 equation, Physica. A., 473(1), 352–362, 2017.

[35]. R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math., 264, 65-70, 2014.

[36]. T. Abdeljawad, On conformable fractional calculus, J. Comput. Appl. Math., 279, 57-66, 2015.

[37]. R. Roy, M. A. Akbar, A. R. Seadawy and D. Baleanu, Search for adequate closed form wave solutions to space-time fractional nonlinear equations, Par. Diff. Equ. App. Math., 2021(4), 100025, 2021.

Appendix-A

The solutions of Roy *and his colleagues* [37], investigated the generalized (G'/G)-expansion method for the space-time fractional Phi-four model are scheduled as follows:

$$u_1 = \pm\sqrt{\frac{\rho}{Q^2+4S\psi}}\tanh\left(\frac{\sqrt{\rho}}{2\psi}\xi\right) \tag{A.1}$$

$$u_2 = \pm\sqrt{\frac{\rho}{Q^2+4S\psi}}coth\left(\frac{\sqrt{\rho}}{2\psi}\xi\right) \tag{A.2}$$

$$u_3 = \mp i\sqrt{\frac{\rho}{Q^2+4S\psi}}tan\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right) \tag{A.3}$$

$$u_4 = \pm i\sqrt{\frac{\rho}{Q^2+4S\psi}}cot\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right) \tag{A.4}$$

$$u_5 = \pm\frac{2\psi}{\sqrt{Q^2+4S\psi}}\left(\frac{C_{22}}{C_{11}+C_{22}\xi}\right) \tag{A.5}$$

$$u_6 = \pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-Q+2\sqrt{\Delta}tanh\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)\right\} \tag{A.6}$$

$$u_7 = \pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-Q+2\sqrt{\Delta}coth\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)\right\} \tag{A.7}$$

$$u_8 = \pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-Q-2i\sqrt{\Delta}tan\left(\frac{\sqrt{-\Delta}}{\psi}\xi\right)\right\} \tag{A.8}$$

$$u_9 = \pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-Q+2i\sqrt{\Delta}cot\left(\frac{\sqrt{-\Delta}}{\psi}\xi\right)\right\} \tag{A.9}$$