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Some New Identities with Respect to Bihyperbolic Fibonacci and Lucas Numbers

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Abstract

In this paper, we define bihyperbolic Lucas numbers, bihyperbolic generalized Fibonacci numbers and give some algebraic properties of these numbers. Then, some identities concerning conjugations, Honsberger's identity, negabihyperbolic numbers for bihyperbolic Fibonacci number and bihyperbolic Lucas numbers have been derived. Finally, well-known identities have been represented such as Cassini's, Catalan's, d'Ocagne's identities and Binet formula for bihyperbolic Lucas and bihyperbolic generalized Fibonacci numbers. Also, special cases of these identities and formulas have been given.

Keywords: Bihyperbolic numbers; generalized bihyperbolic Fibonacci numbers; bihyperbolic Lucas numbers; hyperbolic four-complex numbers.

1. Introduction

Hamilton wanted to show the points in space with their coordinates. Thus, he discovered a new number system. Although this number system which was defined by him extended the complex numbers, the multiplication of these numbers has no commutative property [1]. Then, Cockle revealed the tessarine numbers. The set of tessarine numbers were represented as [2].

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$$T = \left\{ t = a + ib + jc + kd \,\middle|\, a, b, c, d \in R \right\}$$

and multiplication of units i, j, k are as follows

$$i^2 = -1, \ j^2 = 1, k^2 = -1, ij = ji = k.$$

Although the study of repeating the process of building complex numbers upon themselves reaches us tessarines and quaternions, there is an important difference. This difference arises from the fact that the tessarines have commutative property according to the multiplication. Therefore, they can be considered as a kind of commutative quaternions. Furthermore, all nonzero tessarines haven't got inverses.

After Cockle's work on tessarines, Segre has defined bicomplex numbers as follows:

$$BC = \left\{ q = q_1 + iq_2 + jq_3 + kq_4 \,\middle|\, q_1, q_2, q_3, q_4 \in R \right\}$$

where $i^2 = j^2 = -1, k^2 = 1, ij = ji = k$ and the algebra of bicomplex numbers are isomorphic to the Tessarines [3]

The set of bihyperbolic numbers are defined by

$$H_2 = \left\{ q = a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3 \,\middle|\, a_0, a_1, a_2, a_3 \in R; \ j_1, j_2, j_3 \notin R \right\}$$

where the multiplication of the units is given by

$$j_1^2 = j_2^2 = j_3^2 = 1, \qquad j_1 j_2 = j_2 j_1 = j_3, \quad j_1 j_3 = j_3 j_1 = j_2, \quad j_2 j_3 = j_3 j_2 = j_1.$$

Bihyperbolic numbers are also called as hyperbolic four complex numbers [4].

Let $q = a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3$ and $r = b_0 + b_1 j_1 + b_2 j_2 + b_3 j_3$ be two bihyperbolic numbers. Then, the addition and subtraction of these two bihyperbolic numbers can be expressed as follows:

$$q \pm r = a_0 \pm b_0 + (a_1 \pm b_1) j_1 + (a_2 \pm b_2) j_2 + (a_3 \pm b_3) j_3$$

Besides, the multiplication of these numbers are given by

$$\begin{aligned} q \times r = & \left(a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3\right) \\ & + j_1 \left(a_0 b_1 + a_1 b_0 + a_2 b_3 + a_3 b_2\right) \\ & + j_2 \left(a_0 b_2 + a_1 b_3 + a_2 b_0 + a_3 b_1\right) \\ & + j_3 \left(a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0\right). \end{aligned}$$

Bihyperbolic numbers have three different principle conjugations. For any bihyperbolic number $q = a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3$, these conjugations are represented as follows [5]:

$$\begin{aligned} \overline{q}^{j_1} &= a_0 + a_1 j_1 - a_2 j_2 - a_3 j_3 \\ \overline{q}^{j_2} &= a_0 - a_1 j_1 + a_2 j_2 - a_3 j_3 \\ \overline{q}^{j_3} &= a_0 - a_1 j_1 - a_2 j_2 + a_3 j_3. \end{aligned}$$

Thus, the set of commutative tetranumbers can be considered in two parts as bicomplex and bihyperbolic numbers such that bicomplex numbers are the expansion of complex numbers, while bihyperbolic numbers appear as the expansion of hyperbolic numbers into the fourth dimension. Many studies have been done in the fields of functional analysis, topology, algebra and quantum mechanics regarding bicomplex numbers [6-18]. On the other hand, Nurkan and Güven have defined bicomplex Fibonacci and Lucas numbers. Then, they have obtained d'Ocagnes, Cassini, Catalan identities, Binet formulas and also some identities belonging to these new numbers [19]. In [20], Bicomplex Fibonacci quaternions are introduced. Moreover, some well-known identities, properties of these quaternions have been studied. Finally, a real representation of bicomplex Fibonacci quaternions has been given. Later, Halıcı and Çürük have introduced a new bicomplex number sequence by taking the coefficients of bicomplex number sequence as complex numbers. Also, they have found Binet formula, generating function, Cassini, Catalan, Honsberger, d'Ocagne's, Vajda and Gelin-Cesaro identities [21]. Halıcı has obtained identities regarding Fibonacci numbers by taking the idempotent representation of bicomplex numbers. Then, the generalization of these numbers have been made for this new number system and have been called as Horadam bicomplex numbers. Furthermore, the matrix representation have been given for two important identities [22]. Pogoriu and his colleagues has studied the roots of bihyperbolic polynomials [23]. After that, a partial order of bihyperbolic numbers has been represented concerning the spectral representation of bihyperbolic numbers. Also, conjugates, hyperbolic and real-valued modules, absolute value multiplicative inverse and polar form of these numbers have been introduced [24]. Gürses and and his colleagues have studied dual-generalized complex and hyperbolic generalized complex numbers. They have examined the functions and matrix representations of these numbers. For $J = j$ and $p = 1$, they have obtained bihyperbolic numbers [25]. Brod and his colleagues have introduced bihyperbolic Fibonacci, Jacosthal, Pell numbers. Then, they have given well-known identities and summation formulas for these numbers [26]. This work aims to present and extend a new Fibonacci number system which has been firstly described by Brod and his colleagues For this purpose, firstly the definition of the bihyperbolic Lucas numbers and generalized Fibonacci numbers have been given and some algebraic properties have been presented. Afterward, some identities have been described by using three types of conjugates of these numbers. Then, some well-known identities and formula have been established such as negabihyperbolic Fibonacci, identities regarding conjugates and Honsberger identity for bihyperbolic

Fibonacci numbers and d'Ocagne, Cassini, Catalan identities and Binet formula for bihyperbolic Lucas and generalized Fibonacci numbers.

2. Preliminaries

Let W_n be n -th generalized Fibonacci number which has the recurrence relation

$$W_n = W_{n-1} + W_{n-2}; \quad W_0 = a, \quad W_1 = b \quad (n \geq 2)$$

with the nonzero initial values W_0, W_1. If $W_0 = 0, \quad W_1 = 1$, then we obtain Fibonacci numbers and if $W_0 = 2, \quad W_1 = 1$, then we obtain Lucas numbers [27].

The well-known Binet formulas for Lucas and Fibonacci numbers are given by

$$L_n = \alpha^n + \beta^n, \qquad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where α and β are the roots of the equation

$$x^2 - x - 1 = 0$$

so that $\alpha + \beta = 1, \ \alpha\beta = -1, \ \alpha - \beta = \sqrt{5}.$ Also, Binet formula for generalized Fibonacci numbers is

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}$$

where $A = W_1 - W_0\beta$ and $B = W_1 - W_0\alpha$ [27]. Furthermore,

$$A\alpha^n = \alpha W_n + W_{n-1} \tag{1}$$

$$B\beta^n = \beta W_n + W_{n-1} \tag{2}$$

The set of bihyperbolic Fibonacci numbers is defined by

$$H_2^F = \left\{ BHF_n = F_n + j_1 F_{n+1} + j_2 F_{n+2} + j_3 F_{n+3} \middle| F_n \text{ is } n-th \text{ Fibonacci number} \right\}$$

where

$$j_1^2 = j_2^2 = j_3^2 = 1, \qquad j_1 j_2 = j_2 j_1 = j_3, \quad j_1 j_3 = j_3 j_1 = j_2, \quad j_2 j_3 = j_3 j_2 = j_1.$$

Here $j_s\left(s=1,2,3\right)$ denote unipotent(hyperbolic) imaginary units such that $j_s \neq \pm 1$ and $j_s \notin R$, [26]. Furthermore, the addition and multiplication on H_2^F satisfy the commutativeand associative property [26].

Let BHF_n and BHF_m be bihyperbolic Fibonacci numbers These numbers are represented as

$$BHF_n = F_n + j_1F_{n+1} + j_2F_{n+2} + j_3F_{n+3}$$

and

$$BHF_m = F_m + j_1F_{m+1} + j_2F_{m+2} + j_3F_{m+3}$$

respectively. So, the addition, subtraction and multiplication of two bihyperbolic numbers are defined by

$$BHF_n \pm BHF_m = \left(F_n + j_1F_{n+1} + j_2F_{n+2} + j_3F_{n+3}\right) \pm \left(F_m + j_1F_{m+1} + j_2F_{m+2} + j_3F_{m+3}\right). \tag{3}$$

and

$$\begin{aligned} BHF_n \times BHF_m = &\left(F_n + j_1F_{n+1} + j_2F_{n+2} + j_3F_{n+3}\right) \times \left(F_m + j_1F_{m+1} + j_2F_{m+2} + j_3F_{m+3}\right) \\ &= F_nF_m + F_{n+1}F_{m+1} + F_{n+2}F_{m+2} + F_{n+3}F_{m+3} \\ &\quad + j_1\left(F_{n+1}F_m + F_nF_{m+1} + F_{n+3}F_{m+2} + F_{n+2}F_{m+3}\right) \\ &\quad + j_2\left(F_{n+2}F_m + F_nF_{m+2} + F_{n+3}F_{m+1} + F_{n+1}F_{m+3}\right) \\ &\quad + j_3\left(F_{n+3}F_m + F_nF_{m+3} + F_{n+1}F_{m+2} + F_{n+2}F_{m+1}\right) \\ &= BHF_m \times BHF_n. \end{aligned} \tag{4}$$

respectively, [26].

3. Properties of Bihyperbolic Fibonacci and Bihyperbolic Lucas Numbers

In this section, we have introduced the conjugates of any bihyperbolic number. Then, some identities which are related to these conjugates and Honsberger identity have been given. Moreover, negabihyperbolic Fibonacci, negabihyperbolic Lucas numbers, Binet, Catalan, Cassini identities for bihyperbolic Lucas numbers have been proven.

Definition 3.1. Let BHF_n be bihyperbolic Fibonacci number. Then, there exist three different conjugates of any bihyperbolic Fibonacci number such that

$$\begin{aligned}
\overline{BHF}_n^{\,j_1} &= F_n + j_1F_{n+1} - j_2F_{n+2} - j_3F_{n+3} \\
\overline{BHF}_n^{\,j_2} &= F_n - j_1F_{n+1} + j_2F_{n+2} - j_3F_{n+3} \qquad (5)\\
\overline{BHF}_n^{\,j_3} &= F_n - j_1F_{n+1} - j_2F_{n+2} + j_3F_{n+3}.
\end{aligned}$$

Theorem 3.1: Let BHF_n and BHF_m be two bihyperbolic Fibonacci numbers. Then, the following relations can be given between the conjugates of these numbers:

i) $\left(\overline{BHF_n \times BHF_m}\right)^{j_1} = \overline{BHF}_n^{\,j_1} \times \overline{BHF}_m^{\,j_1} = \overline{BHF}_m^{\,j_1} \times \overline{BHF}_n^{\,j_1}$

ii) $\left(\overline{BHF_n \times BHF_m}\right)^{j_2} = \overline{BHF}_n^{\,j_2} \times \overline{BHF}_m^{\,j_2} = \overline{BHF}_m^{\,j_2} \times \overline{BHF}_n^{\,j_2}$

iii) $\left(\overline{BHF_n \times BHF_m}\right)^{j_3} = \overline{BHF}_n^{\,j_3} \times \overline{BHF}_m^{\,j_3} = \overline{BHF}_m^{\,j_3} \times \overline{BHF}_n^{\,j_3}$

Proof. The proof can be easily seen from the equations (4) and (5).

Theorem 3.2: Let BHF_n be bihyperbolic Fibonacci number and $\overline{BHF}_n^{\,j_1}$, $\overline{BHF}_n^{\,j_2}$, $\overline{BHF}_n^{\,j_3}$ be three different conjugations of this number. In this case, the following equations are given between the multiplication of the bihyperbolic number BHF_n and their different conjugates.

i) $BHF_n \times \overline{BHF}_n^{\,j_1} = F_{2n+1} - F_{2n+5} - 2j_1F_{2n+3}$

ii) $BHF_n \times \overline{BHF}_n^{\,j_2} = -F_{2n+3} - 2F_{n+2}F_{n-1} - 2j_2\left(F_{n+1}^2 + F_{n+2}F_{n-1}\right)$

iii) $BHF_n \times \overline{BHF}_n^{\,j_3} = F_{2n+3} + 2j_3(-1)^n$

Proof. i) Considering the equations (4), (5) and using the identity $F_n^2 + F_{n+1}^2 = F_{2n+1}$ [28]

$$\begin{aligned}
BHF_n \times \overline{BHF}_n^{\,j_1} &= \left[\left(F_n^2 + F_{n+1}^2\right) - \left(F_{n+2}^2 + F_{n+3}^2\right)\right] \\
&\quad + 2j_1\left[F_n\, F_{n+1} - \left(F_{n+1} + F_n\right)\left(F_{n+2} + F_{n+1}\right)\right] \\
&= F_{2n+1} - F_{2n+5} + 2j_1\left(-F_{n+2}^2 - F_{n+1}^2\right) \\
&= F_{2n+1} - F_{2n+5} - 2j_1\, F_{2n+3}.
\end{aligned}$$

ii) From the equations (4), (5) and the identities $F_{n+1}^2 - F_n^2 = F_{n+2}\, F_{n-1}$ [29], $F_n\, F_m + F_{n+1}\, F_{m+1} = F_{n+m+1}$

[28], we get

$$\begin{aligned}
BHF_n \times \overline{BHF}_n^{\,j_2} &= -\left(F_{n+1}^2 - F_n^2\right) - \left(F_{n+3}^2 - F_{n+2}^2\right) + 2j_2\left(F_n F_{n+2} - F_{n+1}F_{n+3}\right) \\
&= -F_{n+2}F_{n-1} - F_{n+4}F_{n+1} + 2j_2\left(F_n F_{n+2} - F_{n+1}F_{n+3}\right) \\
&= -F_{n+2}F_{n-1} - \left(F_{n+3} + F_{n+2}\right)F_{n+1} + 2j_2\left(F_n^2 - 2F_{n+1}^2\right) \\
&= -F_{n+2}F_{n-1} - F_{n+3}F_{n+1} - F_{n+2}F_{n+1} - 2j_2\left(F_{n+1}^2 + F_{n+2}F_{n-1}\right) \\
&= -F_{n+2}F_{n-1} - F_{n+3}F_{n+1} - F_{n+2}\left(F_n + F_{n-1}\right) - 2j_2\left(F_{n+1}^2 + F_{n+2}F_{n-1}\right) \\
&= -\left(F_{n+3}F_{n+1} + F_{n+2}F_n\right) - 2F_{n+2}F_{n-1} - 2j_2\left(F_{n+1}^2 + F_{n+2}F_{n-1}\right) \\
&= -F_{2n+3} - 2F_{n+2}F_{n-1} - 2j_2\left(F_{n+1}^2 + F_{n+2}F_{n-1}\right)
\end{aligned}$$

iii) Applying the equations (4), (5) and using the identities $F_{n+1}^2 - F_n^2 = F_{n+2}\,F_{n-1}$ [29], $F_n\,F_m + F_{n+1}\,F_{m+1} = F_{n+m+1}$ [28], $F_m F_{n+1} - F_{m+1}F_n = \left(-1\right)^n F_{m-n}$ [30] and $F_{-n} = \left(-1\right)^{n+1} F_n$ [29,31], we have

$$\begin{aligned}
BHF_n \times \overline{BHF}_n^{\,j_3} &= F_n^2 - F_{n+1}^2 - F_{n+2}^2 + F_{n+3}^2 + 2j_3\left(F_n F_{n+3} - F_{n+1}F_{n+2}\right) \\
&= -F_{n-1}F_{n+2} + F_{n+1}F_{n+4} + 2j_3\left(-1\right)^{n+2} F_{-2} \\
&= -F_{n-1}F_{n+2} + F_{n+1}F_{n+3} + F_{n+1}F_{n+2} - 2j_3\left(-1\right)^n \\
&= -F_{n-1}F_{n+2} + F_{n+3}F_{n+1} + \left(F_n + F_{n-1}\right)F_{n+2} - 2j_3\left(-1\right)^n \\
&= \left(F_n F_{n+2} + F_{n+1}F_{n+3}\right) - 2j_3\left(-1\right)^n \\
&= F_{2n+3} - 2j_3\left(-1\right)^n
\end{aligned}$$

Theorem 3.3 (Honsberger's Identity). Let BHF_n and BHF_m be bihyperbolic Fibonacci numbers. Then, for $n, m \geq 0$ Honsberger Identity for bihyperbolic Fibonacci numbers is given by

$$\begin{aligned}
BHF_n \times BHF_m + BHF_{n+1} \times BHF_{m+1} &= 2\,BHF_{\,n+m+1} + F_{n+m+2} + F_{n+m+5} + F_{n+m+7} \\
&\quad + 2\left(j_1 F_{n+m+6} + j_2 F_{n+m+5} + j_3 F_{n+m+4}\right).
\end{aligned}$$

Proof. From the equations (3) and (4), the following equality is found

$$
\begin{aligned}
BHF_n \times BHF_m + BHF_{n+1} \times BHF_{m+1} = {} & \left(F_n F_m + F_{n+1} F_{m+1}\right) + \left(F_{n+2} F_{m+2} + F_{n+3} F_{m+3}\right) \\
& + \left(F_{n+1} F_{m+1} + F_{n+2} F_{m+2}\right) + \left(F_{n+3} F_{m+3} + F_{n+4} F_{m+4}\right) \\
& + j_1 \Big[\left(F_{n+1} F_m + F_{n+2} F_{m+1}\right) \\
& \qquad + \left(F_n F_{m+1} + F_{n+1} F_{m+2}\right) \\
& \qquad + \left(F_{n+3} F_{m+2} + F_{n+4} F_{m+3}\right) \\
& \qquad + \left(F_{n+2} F_{m+3} + F_{n+3} F_{m+4}\right) \Big] \\
& + j_2 \Big[\left(F_{n+2} F_m + F_{n+3} F_{m+1}\right) \\
& \qquad + \left(F_n F_{m+2} + F_{n+1} F_{m+3}\right) \\
& \qquad + \left(F_{n+3} F_{m+1} + F_{n+4} F_{m+2}\right) \\
& \qquad + \left(F_{n+1} F_{m+3} + F_{n+2} F_{m+4}\right) \Big] \\
& + j_3 \Big[\left(F_{n+3} F_m + F_{n+4} F_{m+1}\right) \\
& \qquad + \left(F_{n+2} F_{m+1} + F_{n+3} F_{m+2}\right) \\
& \qquad + \left(F_{n+1} F_{m+2} + F_{n+2} F_{m+3}\right) \\
& \qquad + \left(F_n F_{m+3} + F_{n+1} F_{m+4}\right) \Big].
\end{aligned}
$$

If the identity $F_n\, F_m + F_{n+1}\, F_{m+1} = F_{n+m+1}$ [28] is considered in the above equality, then the equality becomes

$$
\begin{aligned}
BHF_n \times BHF_m + BHF_{n+1} \times BHF_{m+1} = {} & F_{n+m+1} + F_{n+m+3} + F_{n+m+5} + F_{n+m+7} \\
& + 2 j_1 \left(F_{n+m+2} + F_{n+m+6}\right) + 2 j_2 \left(F_{n+m+3} + F_{n+m+5}\right) \\
& + 4 j_3 F_{n+m+4}
\end{aligned}
$$

When the final equation is arranged, the desired result is easily achieved

$$
\begin{aligned}
BHF_n \times BHF_m + BHF_{n+1} \times BHF_{m+1} = {} & 2\left(F_{n+m+1} + j_1 F_{n+m+2} + j_2 F_{n+m+3} + j_3 F_{n+m+4}\right) \\
& + F_{n+m+2} + F_{n+m+5} + F_{n+m+7} \\
& + 2\left(j_1 F_{n+m+6} + j_2 F_{n+m+5} + j_3 F_{n+m+4}\right) \\
= {} & 2\, BHF_{\,n+m+1} + F_{n+m+2} + F_{n+m+5} + F_{n+m+7} \\
& + 2\left(j_1 F_{n+m+6} + j_2 F_{n+m+5} + j_3 F_{n+m+4}\right).
\end{aligned}
$$

Definition 3.2. Bihyperbolic Lucas number set is defined by

$$
H_2^L = \left\{ BHL_n = L_n + j_1 L_{n+1} + j_2 L_{n+2} + j_3 L_{n+3} \,\middle|\, L_n \text{ is } n-th \text{ Lucas number} \right\}
$$

where

$$
j_1^2 = j_2^2 = j_3^2 = 1, \qquad j_1 j_2 = j_2 j_1 = j_3, \quad j_1 j_3 = j_3 j_1 = j_2, \quad j_2 j_3 = j_3 j_2 = j_1.
$$

Here $j_s\left(s=1,2,3\right)$ denote unipotent(hyperbolic) imaginary units such that $j_s \neq \pm 1$ and $j_s \notin R$.

Theorem 3.4 (Negabihyperbolic Fibonacci and Negahyperbolic Lucas).

Let BHF_n and BHL_n be bihyperbolic Fibonacci and bihyperbolic Lucas number, respectively. For $n \geq 0$, the identities for negabihyperbolic Fibonacci number and negabihyperbolic Lucas number are given by

i) $BHF_{-n} = -\left(-1\right)^n \left[BHF_n + L_n\left(j_1 + j_2 + 2j_3\right)\right]$

ii) $BHL_{-n} = \left(-1\right)^n \left[BHL_n - 5F_n\left(j_1 + j_2 + 2j_3\right)\right].$

Proof: Considering the bihyperbolic Fibonacci number BHF_n, the negebihyperbolic Fibonacci number can be written as

$$BHF_{-n} = F_{-n} + j_1F_{-n+1} + j_2F_{-n+2} + j_3F_{-n+3}.$$

Now, if we use the identity $F_{-n} = \left(-1\right)^{n+1} F_n$ [29,31] and adding and subtracting the terms to the equality, we get

$$\begin{aligned} BHF_{-n} = & -\left(-1\right)^n F_n + j_1\left(-1\right)^n F_{n-1} - j_2\left(-1\right)^n F_{n-2} + j_3\left(-1\right)^n F_{n-3} \\ & + j_1\left(-1\right)^n F_{n+1} - j_1\left(-1\right)^n F_{n+1} + j_2\left(-1\right)^n F_{n+2} - \left(-1\right)^n F_{n+2} + j_3\left(-1\right)^n F_{n+3} - j_3\left(-1\right)^n F_{n+3} \\ = & -\left(-1\right)^n \left(F_n + j_1F_{n+1} + j_2F_{n+2} + j_3F_{n+3}\right) \\ & + \left(-1\right)^n \left[j_1\left(F_{n-1} + F_{n+1}\right) + j_2\left(-F_{n-2} + F_{n+2}\right) + j_3\left(F_{n-3} + F_{n+3}\right)\right]. \end{aligned}$$

Finally, if we use the identities, $F_{n-1} + F_{n+1} = L_n$, $F_{n+2} - F_{n-2} = L_n$, $F_{n+3} + F_{n-3} = 2L_n$ [28], we find

$$\begin{aligned} BHF_{-n} &= -\left(-1\right)^n BHF_n + \left(-1\right)^n \left(j_1L_n + j_2L_n + 2j_3L_n\right) \\ BHF_{-n} &= -\left(-1\right)^n \left[BHF_n + L_n\left(j_1 + j_2 + 2j_3\right)\right]. \end{aligned}$$

ii) Considering the bihyperbolic Lucas number BHL_n, the negebihyperbolic Lucas number can be written as

$$BHL_{-n} = L_{-n} + j_1L_{-n+1} + j_2L_{-n+2} + j_3L_{-n+3}.$$

From the identities $L_{-n} = \left(-1\right)^n L_n$ [29,31] and $L_{n-1} + L_{n+1} = 5F_n$ [32] gives us

$$
\begin{aligned}
BHL_{-n} &= (-1)^n \left(L_n + j_1 L_{n+1} + j_2 L_{n+2} + j_3 L_{n+3}\right) \\
&\quad -(-1)^n j_1 \left(L_{n-1} + L_{n+1}\right) + (-1)^n j_2 \left(L_{n-2} - L_{n+2}\right) - (-1)^n j_3 \left(L_{n-3} + L_{n+3}\right) \\
&= (-1)^n \left[BHL_n - 5 j_1 F_n - 5 j_2 F_n - j_3 \left(5F_{n-2} + \left(L_n + L_{n+1}\right) + L_n\right)\right] \\
&= (-1)^n \left[BHL_n - 5 j_1 F_n - 5 j_2 F_n - j_3 \left(5F_{n-2} + \left(L_{n+2} + L_n\right)\right)\right] \\
&= (-1)^n \left[BHL_n - 5 j_1 F_n - 5 j_2 F_n - 5 j_3 \left(F_{n-2} + F_{n+1}\right)\right] \\
&= (-1)^n \left[BHL_n - 5 j_1 F_n - 5 j_2 F_n - 5 j_3 \left(F_{n-2} + F_{n+1}\right)\right] \\
&= (-1)^n \left[BHL_n - 5F_n \left(j_1 + j_2 + 2 j_3\right)\right].
\end{aligned}
$$

Let BHL_n and BHL_m be two elements in BH_L. Then, the following equalities can be written

$$BHL_n = L_n + L_{n+1}\, j_1 + L_{n+2}\, j_2 + L_{n+3}\, j_3 \tag{6}$$

and

$$BHL_m = L_m + L_{m+1}\, j_1 + L_{m+2}\, j_2 + L_{m+3}\, j_3. \tag{7}$$

Thus, using (6) and (7) the addition, substraction, multiplication and conjugates are given by

$$BHL_n \pm BHL_m = \left(L_n + j_1 L_{n+1} + j_2 L_{n+2} + j_3 L_{n+3}\right) \pm \left(L_m + j_1 L_{m+1} + j_2 L_{m+2} + j_3 L_{m+3}\right) \tag{8}$$

$$
\begin{aligned}
BHL_n \times BHL_m &= \left(L_n + j_1 L_{n+1} + j_2 L_{n+2} + j_3 L_{n+3}\right) \times \left(L_m + j_1 L_{m+1} + j_2 L_{m+2} + j_3 L_{m+3}\right) \\
&= L_n L_m + L_{n+1} L_{m+1} + L_{n+2} L_{m+2} + L_{n+3} L_{m+3} \\
&\quad + j_1 \left(L_{n+1} L_m + L_n L_{m+1} + L_{n+3} L_{m+2} + L_{n+2} L_{m+3}\right) \\
&\quad + j_2 \left(L_{n+2} L_m + L_n L_{m+2} + L_{n+3} L_{m+1} + L_{n+1} L_{m+3}\right) \\
&\quad + j_3 \left(L_{n+3} L_m + L_n L_{m+3} + L_{n+1} L_{m+2} + L_{n+2} L_{m+1}\right) \\
&= BHL_m \times BHL_n.
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
\overline{BHL}_n^{j_1} &= F_n + j_1 F_{n+1} - j_2 F_{n+2} - j_3 F_{n+3} \\
\overline{BHL}_n^{j_2} &= F_n - j_1 F_{n+1} + j_2 F_{n+2} - j_3 F_{n+3} \\
\overline{BHL}_n^{j_3} &= F_n - j_1 F_{n+1} - j_2 F_{n+2} + j_3 F_{n+3}.
\end{aligned} \tag{10}
$$

respectively.

Theorem 3.5: Let BHL_n and BHL_m be two bihyperbolic Lucas numbers. Then, the following relations can be given between the conjugates of these numbers:

i) $\left(\overline{BHL_n \times BHL_m}\right)^{j_1} = \overline{BHL_n}^{j_1} \times \overline{BHL_m}^{j_1} = \overline{BHL_m}^{j_1} \times \overline{BHL_n}^{j_1}$

ii) $\left(\overline{BHL_n \times BHL_m}\right)^{j_2} = \overline{BHL_n}^{j_2} \times \overline{BHL_m}^{j_2} = \overline{BHL_m}^{j_2} \times \overline{BHL_n}^{j_2}$

iii) $\left(\overline{BHL_n \times BHL_m}\right)^{j_3} = \overline{BHL_n}^{j_3} \times \overline{BHL_m}^{j_3} = \overline{BHL_m}^{j_3} \times \overline{BHL_n}^{j_3}$

Proof. The proof is straightforwardly obtained by using the equations (9) and (10).

Theorem 3.6: $\overline{BHL_n}^{j_1}$, $\overline{BHL_n}^{j_2}$, $\overline{BHL_n}^{j_3}$ be three different conjugations of the bihyperbolic number BHL_n. Then, the following equalities hold:

i) $BHL_n \times \overline{BHL_n}^{j_1} = (-5)\left(L_{2n+3} - 2j_1F_{2n+3}\right)$

ii) $BHL_n \times \overline{BHL_n}^{j_2} = -\left(2L_{n-1}L_{n+2} + 5F_{2n+3}\right) - 2j_2\left(L_{n+1}^2 - L_{n-1}L_{n+2}\right)$

iii) $BHL_n \times \overline{BHL_n}^{j_3} = 5\left(F_{2n+3} + 2j_3(-1)^{-n}\right)$

Proof. i) From the equations (9) and (10), it follows that

$$BHL_n \times \overline{BHL_n}^{j_1} = L_n^2 + L_{n+1}^2 - L_{n+2}^2 - L_{n+3}^2 + 2j_1\left(L_nL_{n+1} - L_{n+2}L_{n+3}\right).$$

Then, using the the identities $L_n^2 + L_{n+1}^2 = 5F_{2n+1}$ [32] and $F_{n+2} - F_{n-2} = L_n$ [32], it is seen that

$$\begin{aligned}
BHL_n \times \overline{BHL_n}^{j_1} &= \left(L_n^2 + L_{n+1}^2\right) - \left(L_{n+2}^2 + L_{n+3}^2\right) + 2j_1\left[L_nL_{n+1} - \left(L_{n+1} + L_n\right)\left(L_{n+2} + L_{n+1}\right)\right] \\
&= 5F_{2n+1} - 5F_{2n+5} + 2j_1\left[-L_{n+1}^2 - L_{n+2}\left(L_{n+1} + L_n\right)\right] \\
&= 5F_{2n+1} - 5F_{2n+5} - 2j_1\left(L_{n+2}^2 + L_{n+1}^2\right) \\
&= 5\left(F_{2n+1} - F_{2n+5}\right) - 10j_1F_{2n+3} \\
&= -5\left(L_{2n+3} + 2j_1F_{2n+3}\right).
\end{aligned}$$

ii) Taking into account the identities (9) and (10), we reach

$$BHL_n \times \overline{BHL_n}^{j_2} = L_n^2 - L_{n+1}^2 + L_{n+2}^2 - L_{n+3}^2 + 2\, j_2 \left(L_n L_{n+2} - L_{n+1} L_{n+3} \right)$$

On the other hand, replacing the identities $L_{n+1}^2 - L_n^2 = L_{n+2}\, L_{n-1}$ [32], $L_n\, L_m + L_{n+1}\, L_{m+1} = 5F_{n+m+1}$ [33] in the above equality, we obtain the following equation

$$\begin{aligned} BHL_n \times \overline{BHL_n}^{j_2} &= \left(L_n^2 - L_{n+1}^2 \right) + \left(L_{n+2}^2 - L_{n+3}^2 \right) + 2\, j_2 \left(L_n L_{n+2} - L_{n+1} L_{n+3} \right) \\ &= -\left(L_{n-1} L_{n+2} + L_{n+1} L_{n+4} \right) + 2\, j_2 \left(L_n L_{n+1} - L_{n+1} L_{n+2} - L_{n+1}^2 \right) \\ &= -\left[L_{n-1} L_{n+2} + L_{n+1} \left(L_{n+3} + L_{n+2} \right) \right] + 2\, j_2 \left[L_n L_{n+1} - L_{n+1} \left(L_{n+1} + L_n \right) - L_{n+1}^2 \right] \\ &= -\left[L_{n-1} L_{n+2} + \left(L_{n+1} L_{n+3} + L_n L_{n+2} \right) + L_{n-1} L_{n+2} \right] - 2\, j_2 \left[L_n L_{n+1} - L_{n+1} \left(L_{n+1} + L_n \right) - L_{n+1}^2 \right] \\ &= -5F_{2n+3} - 2L_{n-1} L_{n+2} - 2\, j_2 \left(L_{n+1}^2 + L_{n-1} L_{n+2} \right). \end{aligned}$$

iii) Again, by using the equations (9) and (10), we have

$$BHL_n \times \overline{BHL_n}^{j_3} = L_n^2 - L_{n+1}^2 - L_{n+2}^2 + L_{n+3}^2 + 2\, j_3 \left(L_n L_{n+3} - L_{n+1} L_{n+2} \right).$$

Now, applying the identities $L_{n+1}^2 - L_n^2 = L_{n+2}\, L_{n-1}$ [32], $L_n\, L_m + L_{n+1}\, L_{m+1} = 5F_{n+m+1}$ [33], $\left(-1\right)^{n-1} \left(L_m L_{n+1} - L_{m+1} L_n \right) = 5F_{m-n}$ [33] and $F_{-n} = \left(-1\right)^{n+1} F_n$ [29,31], we can get

$$\begin{aligned} BHL_n \times \overline{BHL_n}^{j_3} &= -\left(L_{n+1}^2 - L_n^2 \right) - \left(L_{n+3}^2 - L_{n+2}^2 \right) + 2\, j_3 \left(L_n L_{n+3} - L_{n+1} L_{n+2} \right) \\ &= -L_{n-1} L_{n+2} + L_{n+1} L_{n+4} - 10\, j_3 \left(-1\right)^{-n} F_{-2} \\ &= -L_{n-1} L_{n+2} + L_{n+1} L_{n+3} + L_{n+1} L_{n+2} + 10\, j_3 \left(-1\right)^{-n} \\ &= -L_{n-1} L_{n+2} + L_{n+3} L_{n+1} + \left(L_n + L_{n-1} \right) L_{n+2} + 10\, j_3 \left(-1\right)^{-n} \\ &= \left(L_n L_{n+2} + L_{n+1} L_{n+3} \right) + 10\, j_3 \left(-1\right)^{-n} \\ &= 5F_{2n+3} + 10\, j_3 \left(-1\right)^{n}. \end{aligned}$$

Theorem 3.7 Let BHL_n and BHL_m be any elements of bihyperbolic Lucas and bihyperbolic Fibonacci number set, respectively. Then, there are relationships between these numbers and their conjugates as follows:

i) $BHF_n + \overline{BHF}_n^{j_1} + \overline{BHF}_n^{j_2} + \overline{BHF}_n^{j_3} = 4F_n$

ii) $BHL_n + \overline{BHL}_n^{j_1} + \overline{BHL}_n^{j_2} + \overline{BHL}_n^{j_3} = 4L_n.$

Proof. From the equations (8) and (10), we get

$$BHF_n + \overline{BHF}_n^{j_1} = 2\left(F_n + j_1 F_{n+1}\right)$$
$$BHF_n + \overline{BHF}_n^{j_2} = 2\left(F_n + j_2 F_{n+2}\right)$$
$$BHF_n + \overline{BHF}_n^{j_3} = 2\left(F_n + j_3 F_{n+3}\right)$$

If the three equalities which is obtained above add side by side with each other, we get

$$\begin{aligned}
BHF_n + \overline{BHF}_n^{j_1} + BHF_n + \overline{BHF}_n^{j_2} + BHF_n + \overline{BHF}_n^{j_3} &= 2\left(F_n + j_1 F_{n+1}\right) + 2\left(F_n + j_2 F_{n+2}\right) + 2\left(F_n + j_3 F_{n+3}\right) \\
3BHF_n + \overline{BHF}_n^{j_1} + \overline{BHF}_n^{j_2} + \overline{BHF}_n^{j_3} &= 2\left(BHF_n + 2F_n\right) \\
BHF_n + \overline{BHF}_n^{j_1} + \overline{BHF}_n^{j_2} + \overline{BHF}_n^{j_3} &= 4F_n.
\end{aligned}$$

The proof of **ii)** is analogous.

Theorem 3.8 There exist the following identities between bihyperbolic Fibonacci and bihyperbolic Lucas numbers.

i) $BHF_n + BHL_n = 2BHF_{n+1}$

ii) $BHL_{n-1} + BHL_{n+1} = 5BHF_n.$

Proof.

i) By direct calculations of the equation (3) and the definition of bihyperbolic Fibonacci number yields to

$$\begin{aligned}
BHF_n + BHL_n &= F_n + j_1 F_{n+1} + j_2 F_{n+2} + j_3 F_{n+3} \\
&\quad + L_n + j_1 L_{n+1} + j_2 L_{n+2} + j_3 L_{n+3} \\
&= F_n + j_1 F_{n+1} + j_2 F_{n+2} + j_3 F_{n+3} \\
&\quad + \left(F_{n-1} + F_{n+1}\right) + j_1\left(F_n + F_{n+2}\right) + j_2\left(F_{n+1} + F_{n+3}\right) + j_3 \\
&= F_{n+1} + j_1 F_{n+2} + j_2 F_{n+3} + j_3 F_{n+4} \\
&\quad + \left(F_n + F_{n-1}\right) + j_1\left(F_n + F_{n+1}\right) + j_2\left(F_{n+1} + F_{n+2}\right) + j_3\left(F_{n+2} + F_{n+3}\right) \\
&= BHF_{n+1} + F_{n+1} + j_1 F_{n+2} + j_2 F_{n+3} + j_3 F_{n+4} \\
&= 2BHF_{n+1}.
\end{aligned}$$

ii) Applying the similar method of i), we find

$$\begin{aligned}
BHL_{n-1}+BHL_{n+1} &= L_{n-1}+j_1L_n+j_2L_{n+1}+j_3L_{n+2} \\
&\quad +L_{n+1}+j_1L_{n+2}+j_2L_{n+3}+j_3L_{n+4} \\
&= \left(F_{n-2}+F_{n-1}\right)+4F_n \\
&\quad + j_1\left[\left(F_{n-1}+F_n\right)+4F_{n+1}\right] \\
&\quad + j_2\left[\left(F_n+F_{n+1}\right)+4F_{n+2}\right] \\
&\quad + j_3\left[\left(F_{n+1}+F_{n+2}\right)+4F_{n+3}\right] \\
&= 5\left(F_n+j_1F_{n+1}+j_2F_{n+2}+j_3F_{n+3}\right) \\
&= 5BHF_n.
\end{aligned}$$

Theorem 3.9 (Binet Formula). Let BHL_n be bihyperbolic Lucas number. For $n \geq 1$, Binet Formula is given by

$$BHL_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n$$

such that

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}, \quad \hat{\alpha} = 1+j_1\alpha+j_2\alpha^2+j_3\alpha^3, \hat{\beta} = 1+j_1\beta+j_2\beta^2+j_3\beta^3.,$$

Proof. If the Binet formula of Lucas numbers is written in the equation (6) and rearranged, we get the desired one as follows

$$\begin{aligned}
BHL_n &= \alpha^n+\beta^n+j_1\alpha^{n+1}+\beta^{n+1}+j_2\alpha^{n+2}+\beta^{n+2}+j_3\alpha^{n+3}+\beta^{n+3} \\
&= \left(1+j_1\alpha+j_2\alpha^2+j_3\alpha^3\right)\alpha^n-\left(1+j_1\beta+j_2\beta^2+j_3\beta^3\right)\beta^n \\
&= \hat{\alpha}\alpha^n+\hat{\beta}\beta^n.
\end{aligned}$$

where $\hat{\alpha} = 1+j_1\alpha+j_2\alpha^2+j_3\alpha^3$ and $\hat{\beta} = 1+j_1\beta+j_2\beta^2+j_3\beta^3$.

Now, Catalan and Cassini identities for bihyperbolic Lucas numbers will be obtained by the help of Binet formula of bihyperbolic numbers.

Teorem 3.10 (Catalan's Identity). Let BHL_n, BHL_{n-m} and BHL_{n+m} be bihyperbolic Lucas numbers. For $m \leq n$ and $n \geq 1$, the Catalan's identity for bihyperbolic Lucas numbers is given by

$$BHL_{n+r} \times BHL_{n-r} - BHL_n^2 = (-1)^{n-m}\left(L_{2m}-2(-1)^m\right)\left(2j_1+3j_3\right).$$

Proof. Considering the Theorem 3.6 for the bihyperbolic Lucas numbers BHL_n, BHL_{n-m} and BHL_{n+m}, one can write the following identities

$$BHL_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n, \quad BHL_{n+m} = \hat{\alpha}\alpha^{n+m} + \hat{\beta}\beta^{n+m}, \quad BHL_{n-m} = \hat{\alpha}\alpha^{n-m} + \hat{\beta}\beta^{n-m}$$

Using the above identities, the equation (9) and the identity $L_n = \alpha^n + \beta^n$, we have

$$\begin{aligned} BHL_{n+m} \times BHL_{n-m} - BHL_n^2 &= \left(\hat{\alpha}\alpha^{n+m} + \hat{\beta}\beta^{n+m}\right)\left(\hat{\alpha}\alpha^{n-m} + \hat{\beta}\beta^{n-m}\right) - \left(\hat{\alpha}\alpha^n + \hat{\beta}\beta^n\right)^2 \\ &= \hat{\alpha}\hat{\beta}\left(\alpha^{n-m}\beta^{n+m} + \alpha^{n+m}\beta^{n-m} - 2\alpha^n\beta^n\right) \\ &= \hat{\alpha}\hat{\beta}\,\alpha^{n-m}\beta^{n-m}\,\frac{\left(\alpha^{n-m}\beta^{n+m} + \alpha^{n+m}\beta^{n-m} - 2\alpha^n\beta^n\right)}{\alpha^{n-m}\beta^{n-m}} \\ &= \hat{\alpha}\hat{\beta}\,\alpha^{n-m}\beta^{n-m}\left(\beta^{2m} + \alpha^{2m} - 2(\alpha\beta)^m\right) \\ &= (2j_1 + 3j_3)(-1)^{n-m}\left(L_{2m} - 2(-1)^m\right). \end{aligned}$$

such that

$$\hat{\alpha}\,\hat{\beta} = \left(1 + j_1\alpha + j_2\alpha^2 + j_3\alpha^3\right)\left(1 + j_1\beta + j_2\beta^2 + j_3\beta^3\right) = 2j_1 + 3j_3.$$

Teorem 3.11 (Cassini's Identity). Let BHL_n be bihyperbolic Lucas number. For $n \geq 1$, Cassini's Identity is

$$BHL_{n-1} \times BHL_{n+1} - BHL_n^2 = 5(-1)^{n-1}(2j_1 + 3j_3).$$

Proof. Considering the Theorem 3.6 for the bihyperbolic Lucas numbers BHL_n, BHL_{n-1} and BHL_{n+1}, one can write the following equalities

$$BHL_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n, \quad BHL_{n+1} = \hat{\alpha}\alpha^{n+1} + \hat{\beta}\beta^{n+1}, \quad BHL_{n-1} = \hat{\alpha}\alpha^{n-1} + \hat{\beta}\beta^{n-1}$$

Using the above equalities and the equation (9), we find the desired.

$$
\begin{aligned}
BHL_{n-1} \times BHL_{n+1} - BHL_n^2 &= \left(\hat{\alpha}\alpha^{n+1} + \hat{\beta}\beta^{n+1}\right)\left(\hat{\alpha}\alpha^{n-1} + \hat{\beta}\beta^{n-1}\right) - \left(\hat{\alpha}\alpha^{n} + \hat{\beta}\beta^{n}\right)^2 \\
&= \hat{\alpha}\hat{\beta}\left(\alpha^{n-1}\beta^{n+1} + \alpha^{n+1}\beta^{n-1} - 2\alpha^n\beta^n\right) \\
&= \hat{\alpha}\hat{\beta}\,\alpha^n\beta^n\left(\alpha^{-1}\beta + \alpha\,\beta^{-1} - 2\right) \\
&= \hat{\alpha}\hat{\beta}\,\alpha^n\beta^n\,\frac{\left(\beta^2 + \alpha^2 - 2\alpha\beta\right)}{\alpha\beta} \\
&= 5(-1)^{n-1}\left(2j_1 + 3j_3\right)
\end{aligned}
$$

where $\hat{\alpha}\,\hat{\beta} = 2j_1 + 3j_3$, $\alpha\beta = -1, \quad \alpha^2 + \beta^2 - 2\alpha\beta = 5.$

The same equality can be obtained by writing 1 instead of m in Theorem 3.10.

Teorem 3.12 (d'Ocagne's Identity). Let BHL_n, BHL_{n-m} and BHL_{n+m} be bihyperbolic Lucas numbers. For $n \le m$ and $n \ge 0, \quad m \ge 0$, the following identity holds.

$$BHL_{m+1} \times BHL_n - BHL_m \times BHL_{n+1} = \left(2j_1 + 3j_3\right)\left(L_n L_{m-1} - L_m L_{n-1}\right)$$

Proof. Using Theorem 3.6 and equation (9), we reach

$$
\begin{aligned}
BHL_{m+1} \times BHL_n - BHL_m \times BHL_{n+1} &= \left(\hat{\alpha}\alpha^{m+1} + \hat{\beta}\beta^{m+1}\right)\left(\hat{\alpha}\alpha^{n} + \hat{\beta}\beta^{n}\right) - \left(\hat{\alpha}\alpha^{m} + \hat{\beta}\beta^{m}\right)^2 \\
&= \hat{\alpha}\hat{\beta}\left(\alpha^n\beta^{m+1} + \alpha^{m+1}\beta^n - \alpha^{n+1}\beta^m - \alpha^m\beta^{n+1}\right) \qquad (11) \\
&= \hat{\alpha}\hat{\beta}\left(-\alpha\beta\right)\left(-\alpha^{n-1}\beta^m - \alpha^m\beta^{n-1} + \alpha^n\beta^{m-1} + \alpha^{m-1}\beta^n\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
L_n L_{m-1} - L_m L_{n-1} &= \left(\alpha^n + \beta^n\right)\left(\alpha^{m-1} + \beta^{m-1}\right) - \left(\alpha^m + \beta^m\right)\left(\alpha^{n-1} + \beta^{n-1}\right) \qquad (12) \\
&= -\alpha^{n-1}\beta^m - \alpha^m\beta^{n-1} + \alpha^n\beta^{m-1} + \alpha^{m-1}\beta^n
\end{aligned}
$$

Substituting the equation (12) into the equation (11), we obtain

$$BHL_{m+1} \times BHL_n - BHL_m \times BHL_{n+1} = \left(2j_1 + 3j_3\right)\left(L_n L_{m-1} - L_m L_{n-1}\right).$$

where $\hat{\alpha}\,\hat{\beta} = 2j_1 + 3j_3, \quad \alpha\beta = -1.$

4. Bihyperbolic Generalized Fibonacci Numbers

In this section, we give the definition of bihyperbolic generalized Fibonacci numbers. Then, Binet formula, Catalan, Cassini and d'Ocagne identities are proven for these numbers. Finally, the special cases of these identities will be mentioned.

Definition 4.1.

Let W_n be n-th generalized Fibonacci number. Then, n-th bihyperbolic generalized Fibonacci number is defined by

$$BHW_n = W_n + j_1 W_{n+1} + j_2 W_{n+2} + j_3 W_{n+3} \tag{13}$$

such that their coefficients are generalized Fibonacci numbers. Here, j_1, j_2, j_3 denotes the hyperbolic imaginary units $(j_k^2 = 1, \quad j_k \neq \mp 1, \quad k = 1,2,3)$ and multiplication of base elements are as follows:

$$j_1^2 = j_2^2 = j_3^2 = 1, \qquad j_1 j_2 = j_2 j_1 = j_3, \quad j_1 j_3 = j_3 j_1 = j_2, \quad j_2 j_3 = j_3 j_2 = j_1.$$

Also, bihyperbolic generalized Fibonacci numbers satisfies the recurrence relation such that

$$BHW_n = BHW_{n-1} + BHW_{n-2}$$

Moreover, first two elements of these numbers are given by

$$BHW_0 = a + bj_1 + (a+b)\, j_2 + (a+2b)\, j_3$$
$$BHW_1 = b + (a+b)\, j_1 + (a+2b)\, j_2 + j_3 (2a+3b)$$

respectively.

Teorem 4.1. Binet formula for bihyperbolic generalized Fibonacci number is given by

$$BHW_n = \frac{A\hat{\alpha}\alpha^n - B\hat{\beta}\beta^n}{\alpha - \beta} \tag{14}$$

where $\hat{\alpha} = 1 + \alpha j_1 + \alpha^2 j_2 + \alpha^3 j_3$ and $\hat{\beta} = 1 + \beta j_1 + \beta^2 j_2 + \beta^3 j_3$.

Proof. Let BHW_n be bihyperbolic generalized Fibonacci number. Replacing the Binet formula which is given for generalized Fibonacci numbers, we get

$$BHW_n = \frac{A\left(1+\alpha j_1+\alpha^2 j_2+\alpha^3 j_3\right)\alpha^n - B\left(1+\beta j_1+\beta^2 j_2+\beta^3 j_3\right)}{\alpha-\beta}$$
$$= \frac{A\hat{\alpha}\alpha^n - B\hat{\beta}\beta^n}{\alpha-\beta}.$$

Corollary 4.1: For any integer n, $W_n = F_n, \quad F_0 = 0, \quad F_1 = 1$, Binet formula for bihyperbolic Fibonacci numbers appears as follows

$$BHF_n = \frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha-\beta} \quad \text{(see [26]).}$$

Corollary 4.2: For any integer n, $W_n = L_n, \quad L_0 = 2, \quad L_1 = 1$, Binet formula for bihyperbolic Fibonacci numbers appears as follows

$$BHL_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n \quad \text{(see Theorem 3.9).}$$

Theorem 4.2. For all positive integers n and m and $n \geq m$, Catalan identity for bihyperbolic generalized Fibonacci number is given by

$$BHW_{n+m}BHW_{n-m} - BHW_n^2 = \frac{(-1)^{n-m+1}(2j_1+3j_3)\left[(A+B)W_{2m-1}+(\beta A+\alpha B)W_{2m}-2(-1)^m AB\right]}{5}.$$

Proof. Applying the Binnet formula for generalized Fibonacci number, we have

$$BHW_{n+m}BHW_{n-m} - BHW_n^2 = \frac{\left(A\hat{\alpha}\alpha^{n+m} - B\hat{\beta}\beta^{n+m}\right)\left(A\hat{\alpha}\alpha^{n-m} - B\hat{\beta}\beta^{n-m}\right) - \left(A\hat{\alpha}\alpha^n - B\hat{\beta}\beta^n\right)^2}{(\alpha-\beta)^2}$$
$$= \frac{-AB\hat{\alpha}\hat{\beta}\left(\alpha^{n-m}\beta^{n+m} + \alpha^{n+m}\beta^{n-m} - 2\alpha^n\beta^n\right)}{(\alpha-\beta)^2}$$
$$= \frac{-AB\hat{\alpha}\hat{\beta}\left(\alpha^{2m} + \beta^{2m} - 2\alpha^m\beta^m\right)(\alpha\beta)^n}{(\alpha-\beta)^2(\alpha\beta)^m}$$
$$= \frac{-AB\hat{\alpha}\hat{\beta}\left(\alpha^{2m} + \beta^{2m} - 2\alpha^m\beta^m\right)(\alpha\beta)^n}{(\alpha-\beta)^2(\alpha\beta)^m}$$

Substituting the equations (1), (2) and $\hat{\alpha}\hat{\beta} = 2j_1 + 3j_3$, into the last equality and rearranging, we reach the desired result as follows:

$$BHW_{n+m}BHW_{n-m}-BHW_n^2=\frac{(-1)^{n-m+1}(2j_1+3j_3)\left[B(\alpha W_{2m}+W_{2m-1})+A(\beta W_{2m}+W_{2m-1})-2AB(\alpha\beta)^m\right]}{5}$$
$$=\frac{(-1)^{n-m+1}(2j_1+3j_3)\left[(A+B)W_{2m-1}+(\alpha B+\beta A)W_{2m}-2AB(-1)^m\right]}{5}.$$

Corollary 4.3: For any integer n, $W_n=F_n,\quad F_0=0,\quad F_1=1$, Catalan identity is obtained for bihyperbolic Fibonacci numbers as follows

$$BHF_{n+m}BHF_{n-m}-BHF_n^2=\frac{(-1)^{n-m+1}(2j_1+3j_3)\left[2F_{2m-1}+F_{2m}-2(-1)^m\right]}{5}.$$

If the identity $F_m^2=\frac{2F_{2m-1}+F_{2m}-2(-1)^m}{5}$ is considered, we get

$$BHF_{n+m}BHF_{n-m}-BHF_n^2=(-1)^{n-m+1}(2j_1+3j_3)F_m^2. \quad \text{(see [26]).}$$

Corollary 4.4: For any integer n, $W_n=L_n,\quad L_0=2,\quad L_1=1$, Catalan identity is obtained for bihyperbolic Lucas numbers

$$BHL_{n+m}BHL_{n-m}-BHL_n^2=(-1)^{n-m}(2j_1+3j_3)\left(L_{2m}-2(-1)^m\right) \quad \text{(see Theorem 3.10).}$$

If the identity $L_m^2=L_{2m}+2(-1)^m$ is taken into account, the above identitycan be represented in a different way. Namely,

$$BHL_{n+m}BHL_{n-m}-BHL_n^2=(-1)^{n-m}(2j_1+3j_3)\left(L_m^2-4(-1)^m\right).$$

Theorem 4.3. For $n\geq 1$, Cassini identity for bihyperbolic generalized Fibonacci number holds

$$BHW_{n+1}\,BHW_{n-1}-BHW_n^2=(-1)^n AB(2j_1+3j_3).$$

Proof. Applying the Binnet formula for generalized Fibonacci number, we can deduce that

$$BHW_{n+1}\,BHW_{n-1}-BHW_n^2=\frac{\left(A\hat{\alpha}\alpha^{n+1}-B\,\hat{\beta}\beta^{n+1}\right)\left(A\hat{\alpha}\alpha^{n-1}-B\,\hat{\beta}\beta^{n-1}\right)-\left(A\hat{\alpha}\alpha^{n}-B\,\hat{\beta}\beta^{n}\right)^2}{\left(\alpha-\beta\right)^2}$$

$$=\frac{-AB\hat{\alpha}\hat{\beta}\left(\alpha^{n-1}\beta^{n+1}+\alpha^{n+1}\beta^{n-1}-2\alpha^n\beta^n\right)}{\left(\alpha-\beta\right)^2}$$

$$=\frac{-\left(\alpha\beta\right)^n AB\hat{\alpha}\hat{\beta}\left(\alpha^2+\beta^2-2\alpha\beta\right)}{\left(\alpha-\beta\right)^2\alpha\beta}$$

For $\alpha^2+\beta^2-2\alpha\beta=5,\quad \alpha\beta=-1,\quad \hat{\alpha}\,\hat{\beta}=2j_1+3j_3$, $\left(\alpha-\beta\right)^2=5$, we get

$$BHW_{n+1}\,BHW_{n-1}-BHW_n^2=\left(-1\right)^n AB\left(2j_1+3j_3\right).$$

Corollary 4.5: For any integer n, $W_n=F_n,\quad F_0=0,\quad F_1=1$, Cassini identity is obtained for bihyperbolic Fibonacci numbers as follows

$$BHF_{n+1}BHF_{n-1}-BHF_n^2=\left(-1\right)^n\left(2j_1+3j_3\right) \text{ (see [26]).}$$

Corollary 4.6: For any integer n, $W_n=L_n,\quad L_0=2,\quad L_1=1$, Cassini identity is obtained for bihyperbolic Lucas numbers as follows

$$BHL_{n+1}\,BHL_{n-1}-BHL_n^2=5\left(-1\right)^{n-1}\left(2j_1+3j_3\right) \text{ (see Theorem 3.11).}$$

Theorem 4.4. Let n and m any integers d'Ocagne identity for bihyperbolic generalized Fibonacci number is

$$BHW_{m+1}BHW_n-BHW_mBHW_{n+1}=\left(2j_1+3j_3\right)\left(W_nW_{m-1}-W_{n-1}W_m\right).$$

Proof. From the Binnet formula for generalized Fibonacci number, the following result is straightforwardly obtained.

$$BHW_{m+1}BHW_n - BHW_mBHW_{n+1} = \frac{\left(A\hat{\alpha}\alpha^{m+1} - B\hat{\beta}\beta^{m+1}\right)\left(A\hat{\alpha}\alpha^{n} - B\hat{\beta}\beta^{n}\right) - \left(A\hat{\alpha}\alpha^{m} - B\hat{\beta}\beta^{m}\right)\left(A\hat{\alpha}\alpha^{n+1} - B\hat{\beta}\beta^{n+1}\right)}{(\alpha-\beta)^2}$$

$$= \frac{AB\hat{\alpha}\hat{\beta}\left(-\alpha^n\beta^{m+1} - \alpha^{m+1}\beta^n + \alpha^{n+1}\beta^m + \alpha^m\beta^{n+1}\right)}{(\alpha-\beta)^2}$$

$$= \frac{AB\hat{\alpha}\hat{\beta}\left[(\alpha-\beta)\alpha^n\beta^m - (\alpha-\beta)\alpha^m\beta^n\right]}{(\alpha-\beta)^2}$$

$$= \frac{AB\hat{\alpha}\hat{\beta}^n\left(\alpha^n\beta^m - \alpha^m\beta^n\right)}{(\alpha-\beta)(\alpha\beta)^m}$$

Considering the equations (1), (2) and $\hat{\alpha}\,\hat{\beta} = 2j_1 + 3j_3,$ for the last equation and rearranging the terms, we find the desired.

$$BHW_{m+1}BHW_n - BHW_mBHW_{n+1} = \frac{\hat{\alpha}\hat{\beta}\left[(\alpha W_n + W_{n-1})(\beta W_m + W_{m-1}) - (\alpha W_m + W_{m-1})(\beta W_n + W_{n-1})\right]}{(\alpha-\beta)}$$

$$= \frac{\hat{\alpha}\hat{\beta}(\alpha-\beta)(W_nW_{m-1} - W_{n-1}W_m)}{(\alpha-\beta)}$$

$$= (2j_1 + 3j_3)(W_nW_{m-1} - W_{n-1}W_m).$$

Corollary 4.7: For any integer n, $W_n = F_n, \quad F_0 = 0, \quad F_1 = 1$, d'Ocagne identityfor bihyperbolic Fibonacci numbers becomes

$$BHF_{m+1}BHF_n - BHF_mBHF_{n+1} = (2j_1 + 3j_3)(F_nF_{m-1} - F_{n-1}F_m)$$

On the other hand, if we use d'Ocagne identity $F_m\,F_{n+1} - F_n\,F_{m+1} = (-1)^n\,F_{m-n}$, d'Ocagne identity is obtained for bihyperbolic Fibonacci numbers as follows

$$BHF_{m+1}BHF_n - BHF_mBHF_{n+1} = (-1)^{n-1}(2j_1 + 3j_3)F_{m-n}$$ (see [26]).

Corollary 4.8: For any integer n, $W_n = L_n, \quad L_0 = 2, \quad L_1 = 1$, d'Ocagne identity is obtained for bihyperbolic Lucas numbers

$$BHL_{m+1}BHL_n - BHL_mBHL_{n+1} = (2j_1 + 3j_3)(L_nL_{m-1} - L_{n-1}L_m)$$ (see Theorem 3.12).

5. Conclusion

In this paper, three different conjugates for bihyperbolic Fibonacci numbers have been defined and have been reached to new results regarding these numbers. Correspondences of the Honsberger identity and the negative Fibonacci numbers for bihyperbolic Fibonacci numbers have been found. Then, the concept of bihyperbolic Lucas number have been introduced and some important identities have been obtained. On the other hand, the bihyperbolic Fibonacci number theory have been extended by taking the coefficients of bihyperbolic numbers as generalized Fibonacci numbers. Finally, special cases have been mentioned. Recently, involvement of Fibonacci numbers in the fields of high energy physics, quantum mechanics, cryptology and coding has accelerated the work on these numbers.For this purpose, polynomials, trigonometric identities and matrix representations of Fibonacci numbers are discussed. Therefore, we hope that this new Fibonacci number system and identities that we have found will offer a new perspective to the readers.

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