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Weighted $\alpha\beta$-equistatistical Convergence for Double Sequences of Functions of Two Variables

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Abstract

V. Karakaya and T.A. Chiristi extended the definition of statistical convergence to weighted statistical convergence in [37], using the sequence of real numbers s_k, satisfying some conditions. The modification of this topic was fulfilled in some papers such as [21,30]. It is well known that if $s_k = 1$, for all k, the weighted statistical convergence reduces to statistical convergence. Karakaya and Karasia [38] defined weighted $\alpha\beta$-statistical convergence of order γ, which does not have this property. In this extension for the case $s_k = 1$, for all k, weighted $\alpha\beta$-statistical convergence of order γ does not reduce to $\alpha\beta$-statistical convergence. Later Aktuğlu and Halil introduced a modification in [12] to remove this extension problem. In this paper we introduce weighted $\alpha\beta$-equistatistical convergence of order (γ,η) for double sequences, by using two real sequences p_k and q_j, considering the modified extension with improved method, also we use this definition to prove Korovkin type approximation theorem via weighted $\alpha\beta$-equistatistical convergence of order (γ,η) and weighted $\alpha\beta$-statistical uniform convergence of order (γ,η) for bivariate functions on $[0,\infty)\times[0,\infty)$. Some examples of positive linear operators are constructed to show that, our approximation results work, but its uniform case does not work. Furthermore rate of weighted $\alpha\beta$-equistatistical convergence of order (γ,η) are studied.

Keywords: Double sequences; Statistical convergence; Equistatistical convergence; Rate of convergence; Korovkin type approximation; weighted statistical convergence; Positive Linear Operator.

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1. Introduction

1. Take $\boldsymbol{D} \subseteq \mathbb{N}$, then the real number $\mathbf{0} \leq \boldsymbol{\delta}(\boldsymbol{D}) \leq \mathbf{1}$, which is defined by,

$$\delta(D) = lim_{n\to\infty} \frac{|\{a\in[1,n]: a\in D\}|}{n},$$

in the condition of existence of the limit, is called the density of the subset D. The |I| indicates the cardinality of set I. In [13], Fast used the natural density to define a new type of convergency which is called, statistical convergence and it is a non-trivial extension of ordinary convergence. For any sequence $\boldsymbol{y_k}$ and $\boldsymbol{\varepsilon} > \mathbf{0}$ if $\boldsymbol{\delta}(\boldsymbol{a} \in [\mathbf{1}, \boldsymbol{n}]: |\boldsymbol{x_k} - \boldsymbol{L}| \geq \boldsymbol{\varepsilon}\}) = \mathbf{0}$, then $\boldsymbol{y_k}$ is called statistically convergent to L and it is shown by $\boldsymbol{st} - \boldsymbol{lim} \boldsymbol{y_k} = \boldsymbol{L}$. Pringsheim [1], introduced the limit of real valued double sequences. A real valued double sequence $\boldsymbol{x_{m,n}}$ is called convergent to ''a'' in Pringsheim's sence (P-sence) and shown as $\boldsymbol{P} - \boldsymbol{lim_{n,m}} \boldsymbol{x_{m,n}} = \boldsymbol{a}$, if for every $\boldsymbol{\varepsilon} > \mathbf{0}$ there exist $\boldsymbol{N_\varepsilon} \in \mathbb{N}$ such that,

$$\left|\boldsymbol{x_{m,n}} - \boldsymbol{a}\right| < \boldsymbol{\varepsilon} \quad \forall \boldsymbol{n}, \boldsymbol{m} \geq \boldsymbol{N_\varepsilon}.$$

Let I be a subset of $\mathbb{N} \times \mathbb{N}$, so the density of I is defined as;

$$\boldsymbol{\delta_2}(\boldsymbol{K}) = \boldsymbol{P} - \boldsymbol{lim_{nm}} \frac{\boldsymbol{K}(\boldsymbol{m},\boldsymbol{n})}{\boldsymbol{mn}},$$

where $\boldsymbol{I}(\boldsymbol{n}, \boldsymbol{m}) = |\{(\boldsymbol{j}, \boldsymbol{k}), \mathbf{1} \leq \boldsymbol{j} \leq \boldsymbol{n}, \mathbf{1} \leq \boldsymbol{k} \leq \boldsymbol{m}: |\boldsymbol{x_{n,m}} - \boldsymbol{a}| \geq \boldsymbol{\varepsilon}\}|$.

Weighted statistical convergence was studied in [21,30,37]. A definition of weighted $\boldsymbol{\alpha\beta}$-statistical convergence of order $\boldsymbol{\gamma}$ is considered in [38]. Later modified form of weighted $\boldsymbol{\alpha\beta}$-statistical convergence was introduced by Aktuğlu and Gezer in [12]. In this paper we use the modification to introduce weighted $\boldsymbol{\alpha\beta}$-equistatistical convergence of order $(\boldsymbol{\gamma}, \boldsymbol{\eta})$ for double sequences of functions, which is an extension of $\boldsymbol{\alpha\beta}$-equistatistical convergence of order $\boldsymbol{\gamma}$. Later Korovkin type approximation theorems are proved via weighted $\boldsymbol{\alpha\beta}$-equistatistical convergence and $\boldsymbol{\alpha\beta}$-statistical uniform convergence of order $(\boldsymbol{\gamma}, \boldsymbol{\eta})$ for double sequences of functions of two variables on $\boldsymbol{E} = [\mathbf{0}, \infty) \times [\mathbf{0}, \infty)$. Approximation results are illustrated on some examples of positive linear operators. The last chapter is devoted to the rate of weighted $\boldsymbol{\alpha\beta}$-equistatistical convergence of order $(\boldsymbol{\gamma}, \boldsymbol{\eta})$. Let $\boldsymbol{\Lambda}$ be the set of all pairs such that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are non-decreasing sequences of positive numbers with $\boldsymbol{\beta}(\boldsymbol{n}) \geq \boldsymbol{\alpha}(\boldsymbol{n})$ for all n, and $\boldsymbol{\beta}(\boldsymbol{n}) - \boldsymbol{\alpha}(\boldsymbol{n}) \to \infty$ as $\boldsymbol{n} \to \infty$. For all $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \boldsymbol{\Lambda}$ and $\boldsymbol{\delta}^{\alpha,\beta}(\boldsymbol{I}, \boldsymbol{\gamma})$ is introduced as follows (see [12]),

$$\boldsymbol{\delta}^{\alpha,\beta}(\boldsymbol{I}, \boldsymbol{\gamma}) = \lim_{\boldsymbol{n}\to\infty} \frac{|\{\boldsymbol{k}\in[\boldsymbol{\alpha}(\boldsymbol{n}),\boldsymbol{\beta}(\boldsymbol{n})]: \boldsymbol{k}\in\boldsymbol{I}\}|}{(\boldsymbol{\beta}(\boldsymbol{n})-\boldsymbol{\alpha}(\boldsymbol{n})+\mathbf{1})^{\gamma}}, \qquad (1.1)$$

where $\mathbf{0} < \boldsymbol{\gamma} \leq \mathbf{1}$. It is obvious that if $\boldsymbol{\alpha}(\boldsymbol{n}) = \mathbf{1}$ and $\boldsymbol{\beta}(\boldsymbol{n}) = \boldsymbol{n}$ then $\boldsymbol{\alpha\beta}$-statistical convergence of order $\boldsymbol{\gamma}$ reduces to statistical convergence of order $\boldsymbol{\gamma}$. Some properties of $\boldsymbol{\delta}^{\alpha,\beta}$, which will be used in the rest of the paper, are given in the following lemma (see [2]).

Lemma (1.1) ([12]**)** If $\boldsymbol{I}, \boldsymbol{J} \subseteq \mathbb{N}$ and $\mathbf{0} < \boldsymbol{\gamma} \leq \mathbf{1}$, then for all $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \boldsymbol{\Lambda}$, the following properties are true,

1) If $\boldsymbol{I} = \emptyset$, then $\boldsymbol{\delta}^{\alpha,\beta}(\boldsymbol{I},\boldsymbol{\gamma})$=0.
2) If $\boldsymbol{I} = \mathbb{N}$, then $\boldsymbol{\delta}^{\alpha,\beta}(\boldsymbol{I},\boldsymbol{\gamma})$=1.
3) If $|\boldsymbol{I}| < \infty$, then $\boldsymbol{\delta}^{\alpha,\beta}(\boldsymbol{I},\boldsymbol{\gamma})$=0
4) Obviously for any subsets $\boldsymbol{I} \subset \boldsymbol{J}$, consequences $\boldsymbol{\delta}^{\alpha,\beta}(\boldsymbol{I},\boldsymbol{\gamma}) \le \boldsymbol{\delta}^{\alpha,\beta}(\boldsymbol{J},\boldsymbol{\gamma})$.
5) $\boldsymbol{\delta}^{\alpha,\beta}(\boldsymbol{I} \cup \boldsymbol{J},\boldsymbol{\gamma}) \le \boldsymbol{\delta}^{\alpha,\beta}(\boldsymbol{I},\boldsymbol{\gamma}) + \boldsymbol{\delta}^{\alpha,\beta}(\boldsymbol{J},\boldsymbol{\gamma})$.
6) If $\mathbf{0} < \boldsymbol{\gamma} \le \boldsymbol{\eta} \le \mathbf{1}$, then $\boldsymbol{\delta}^{\alpha,\beta}(\boldsymbol{I},\boldsymbol{\eta}) \le \boldsymbol{\delta}^{\alpha,\beta}(\boldsymbol{I},\boldsymbol{\gamma})$.

The following is the given definition of $\boldsymbol{\alpha\beta}$-statistical convergence of order $\mathbf{0} < \boldsymbol{\gamma} \le \mathbf{1}$ in [11].

Definition (1.2) ([11]) If the sequence $\boldsymbol{x} = \{\boldsymbol{x_k}\}$, $\boldsymbol{k} \in \mathbb{N}$, is called $\boldsymbol{\alpha\beta}$-statistically convergent to L of order $\boldsymbol{\gamma}$ and denoted by $\boldsymbol{st^{\gamma}_{\alpha\beta}} - \boldsymbol{lim_{n\to\infty} x_n} = \boldsymbol{L}$ for any positive $\boldsymbol{\varepsilon}$, if the following holds,

$$\boldsymbol{\delta}^{\alpha,\beta}(\{\boldsymbol{k} \in [\boldsymbol{\alpha(n)},\boldsymbol{\beta(n)}]: |\boldsymbol{x_k} - \boldsymbol{L}| \ge \boldsymbol{\varepsilon}\},\boldsymbol{\gamma}) = \lim_{\boldsymbol{n}\to\infty} \frac{|\{\boldsymbol{k}\in[\boldsymbol{\alpha(n)},\boldsymbol{\beta(n)}]:|\boldsymbol{x_k}-\boldsymbol{L}|\ge\boldsymbol{\varepsilon}\}|}{(\boldsymbol{\beta(n)}-\boldsymbol{\alpha(n)}+\mathbf{1})^{\boldsymbol{\gamma}}} = \mathbf{0}.$$

It is obvious that, taking $\boldsymbol{\gamma} = \mathbf{1}$, in above equation, it gives the definition of the $\boldsymbol{\alpha\beta}$-statistical convergence.

2. Weighted $\boldsymbol{\alpha\beta}$-statistical convergence for double sequences of order $(\boldsymbol{\gamma},\boldsymbol{\eta})$

Let $\boldsymbol{p_n}$ and $\boldsymbol{q_m}$ be any sequences and let,

$$\boldsymbol{P_n} = \sum_{\boldsymbol{k}\in[\boldsymbol{\alpha(n)},\boldsymbol{\beta(n)}]} \boldsymbol{p_k} \to \infty \quad \boldsymbol{as} \quad \boldsymbol{n} \to \infty, \tag{2.1}$$

and

$$\boldsymbol{Q_m} = \sum_{\boldsymbol{j}\in[\boldsymbol{\alpha(m)},\boldsymbol{\beta(m)}]} \boldsymbol{q_j} \to \infty \quad \boldsymbol{as} \quad \boldsymbol{m} \to \infty, \tag{2.2}$$

where $\boldsymbol{n},\boldsymbol{m} \in \mathbb{N}$. Then for any pair $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \boldsymbol{\Lambda}$ define:

$$\boldsymbol{A_n} = \frac{\boldsymbol{\alpha(n)}}{[\boldsymbol{\alpha(n)}]} \sum_{\boldsymbol{k}=\mathbf{1}}^{[\boldsymbol{\alpha(n)}]} \boldsymbol{p_k} \qquad \boldsymbol{C_m} = \frac{\boldsymbol{\alpha(m)}}{[\boldsymbol{\alpha(m)}]} \sum_{\boldsymbol{j}=\mathbf{1}}^{[\boldsymbol{\alpha(m)}]} \boldsymbol{q_j},$$

and

$$\boldsymbol{B_n} = \frac{\boldsymbol{\beta(n)}}{[\boldsymbol{\beta(n)}]} \sum_{\boldsymbol{k}=\mathbf{1}}^{[\boldsymbol{\beta(n)}]} \boldsymbol{p_k} \quad \boldsymbol{D_m} = \frac{\boldsymbol{\beta(m)}}{[\boldsymbol{\beta(m)}]} \sum_{\boldsymbol{j}=\mathbf{1}}^{[\boldsymbol{\beta(m)}]} \boldsymbol{q_j},$$

where [r] is the integer part of r.

Definition (2.1) Let $\boldsymbol{x} = (\boldsymbol{x_{n,m}})$ be any double sequences, so it is called to be weighted $\boldsymbol{\alpha\beta}$-statistically convergent of order $(\boldsymbol{\gamma},\boldsymbol{\eta})$ to L if $\forall \boldsymbol{\varepsilon} > \mathbf{0}$,

$$P - lim_{n,m} \frac{|\{(k,j) \in [C_m, D_m] \times [A_n, B_n]: p_k q_j |x_{k,j} - L| \geq \varepsilon\}|}{(B_n - A_n + 1)^{\eta} (C_m - D_m + 1)^{\gamma}} = 0.$$

Taking $p_k = q_j = 1$ for all k, j=1,2,... , $A_n = \alpha(n)$, $B_n = \beta(n)$, $C_m = \alpha(m)$ and $D_m = \beta(m)$, in the above equation we have,

$$P - lim_{n,m} \frac{|\{(k,j) \in [\alpha(m), \beta(m)] \times [\alpha(n), \beta(n)]: |x_{k,j} - L| \geq \varepsilon\}|}{(\alpha(n) - \beta(n) + 1)^{\eta} (\alpha(m) - \beta(m) + 1)^{\gamma}} = 0,$$

which is the definition of $\alpha\beta$-statistical convergence of order (γ, η) for double sequences. In this section we consider some examples of $\alpha\beta$-statistical convergence, $\alpha\beta$-equistatistical convergence and $\alpha\beta$-statistical uniform convergence and we show their differences.

Definition (2.2) A double sequences of bivariate functions $\{f_{m,n}\}$ on $X^2 \subseteq \mathbb{R} \times \mathbb{R}$ is said to be $\alpha\beta$-statistically pointwise convergent of order (γ, η) to f, if $\forall \varepsilon \geq 0$ and for each $(x, y) \in X^2$,

$$P - lim_{n,m} \frac{|\{(k,j) \in [\alpha(m), \beta(m)] \times [\alpha(n), \beta(n)]: |f_{k,j}(x,y) - f(x,y)| \geq \varepsilon\}|}{(\alpha(n) - \beta(n) + 1)^{\eta} (\alpha(m) - \beta(m) + 1)^{\gamma}} = 0.$$

Then this is shown as $st_{\alpha\beta}^{\gamma\eta} - f_{n,m} \to f$.

Definition (2.3) A double sequences of bivariate functions $\{f_{m,n}\}$ on $X^2 \subseteq \mathbb{R} \times \mathbb{R}$ is said to be $\alpha\beta$-equistatistically convergent of order (γ, η) to f, if $\forall \varepsilon \geq 0$ and for each (x, y) in X^2 the double sequences of real valued functions of two variables,

$$p_{m,n,\varepsilon,\gamma,\eta}(x, y) := \frac{|\{(k,j) \in [\alpha(m), \beta(m)] \times [\alpha(n), \beta(n)]: |f_{k,j}(x,y) - f(x,y)| \geq \varepsilon\}|}{(\alpha(n) - \beta(n) + 1)^{\eta} (\alpha(m) - \beta(m) + 1)^{\gamma}},$$

Converges uniformly to zero function on X^2 i.e. $P - lim_{m,n} \parallel p_{m,n,\varepsilon,\gamma,\eta}(.,.) \parallel_{C(X^2)} = 0$. By the definition we have the following implication $st_{\alpha\beta}^{\gamma\eta} - f_{m,n} \twoheadrightarrow f$, where $\parallel f \parallel_{C(X^2)} = sup_{(x,y) \in X^2} |f(x, y)|$.

Definition (2.4) A double sequences of bivariate functions $\{f_{m,n}\}$ on $X^2 \subseteq \mathbb{R} \times \mathbb{R}$ is said to be $\alpha\beta$-statistically uniform convergent of order (γ, η) to f if $\forall \varepsilon \geq 0$ and for each (x, y) in X^2,

$$P - lim_{n,m} \frac{\left|\left\{(k,j) \in [\alpha(m), \beta(m)] \times [\alpha(n), \beta(n)]: \parallel f_{k,j} - f \parallel_{C(X^2)} \geq \varepsilon\right\}\right|}{(\alpha(n) - \beta(n) + 1)^{\eta} (\alpha(m) - \beta(m) + 1)^{\gamma}} = 0.$$

Then it is shown by $st_{\alpha\beta}^{\gamma\eta} - f_{m,n} \rightrightarrows f$.

Remark (2.5) 1) If $\gamma = \eta = 1$, then weighted $\alpha\beta$-staistical pointwise convergence, weighted $\alpha\beta$-equistaistical convergence and weighted $\alpha\beta$-staistical uniformly convergence of order (γ, η) are called $\alpha\beta$-statistical pointwise convergence, $\alpha\beta$-equistatistical convergence and $\alpha\beta$- statistical uniformly convergence respectively.

2) Clearly for $\mathbf{0 < \eta, \gamma \leq 1}$, $st_{\alpha\beta}^{\gamma\eta} - f_{m,n} \rightrightarrows f \Rightarrow st_{\alpha\beta}^{\gamma\eta} - f_{m,n} \twoheadrightarrow f \Rightarrow st_{\alpha\beta}^{\gamma\eta} - f_{m,n} \to f$.

The following examples show that the inverse of 2) does not hold.

Example (2.6) Let $\boldsymbol{f_{m,n}}: [\mathbf{0}, \infty) \times [\mathbf{0}, \infty) \to \{\mathbf{0}, \mathbf{1}\}$ be the sequence of functions of two variables defined as,

$$\boldsymbol{f_{m,n}(x, y) := \chi_m(x)\chi_n(y)}$$

where $\boldsymbol{\chi_n(x)}$ is characteristic function of x and let f(x, y)=0. Then $st_{\alpha\beta}^{\gamma\eta} - f_{m,n} \to f$ for all $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \boldsymbol{\Lambda}$ and for $\mathbf{0 < \gamma, \eta \leq 1}$. Moreover for a given $\boldsymbol{\varepsilon > 0}$ we have,

$$p_{m,n,\varepsilon,\gamma,\eta}(x, y) := \frac{|\{(k, j) \in [\alpha(m), \beta(m)] \times [\alpha(n), \beta(n)]: |f_{k,j}(x, y) - f(x, y)| \geq \varepsilon\}|}{(\alpha(n) - \beta(n) + 1)^{\eta}(\alpha(m) - \beta(m) + 1)^{\gamma}}$$
$$\leq \frac{1}{(\alpha(n) - \beta(n) + 1)^{\eta}(\alpha(m) - \beta(m) + 1)^{\gamma}} \to 0,$$

as $m, n \to \infty$. This means that, it is $\alpha\beta$-equistatistical convergent to f, but

$$\sup_{(x,y)\in[0,\infty)\times[0,\infty)} |f_{m,n}(x, y)| = 1,$$

is not $\alpha\beta$-statistically uniform convergent to f.

Example (2.7) Lets define the sequence of functions of two variables $f_{m,n}(x, y) = \frac{x^m}{(1+x)^m} \times \frac{y^n}{(1+y)^n}$, where

$f_{m,n}: [0, \infty) \times [0, \infty) \to [0,1)$.

It is clear that $f_{m,n}(x, y)$ is pointwise convergent to $f(x, y) = 0$ in the P-sence, and shown as $f_{m,n} \to f(\alpha\beta - st.)$, where $(\alpha, \beta) \in \Lambda$. If we choose ε=1/4, then for all $k \in [\alpha(m), \beta(m)]$, $j \in [\alpha(n), \beta(n)]$ and

$(x, y) \in (\frac{1}{\sqrt[\beta(m)]{2}-1}, \infty) \times (\frac{1}{\sqrt[\beta(n)]{2}-1}, \infty)$,

we have

$$f_{k,j}(x, y) = (\frac{x}{1+x})^k (\frac{y}{1+y})^j \geq (\frac{1}{\sqrt[\beta(m)]{2}})^k (\frac{1}{\sqrt[\beta(n)]{2}})^j \geq (\frac{1}{\sqrt[\beta(m)]{2}})^{\beta(m)} (\frac{1}{\sqrt[\beta(n)]{2}})^{\beta(n)} \geq \frac{1}{4},$$

which shows that $st_{\alpha,\beta}^{\gamma\eta} - f_{m,n} \twoheadrightarrow f$ does not hold for $0 < \gamma, \eta \leq 1$.

Example (2.8) Consider the following double sequence of functions of two variables

$g_{k,j}: D = [0,\infty) \times [0,\infty) \to \{0,1\}$,

such that for any $n, m \in \mathbb{N}$:

$$g_{(k,j)}(x,y) = \begin{cases} 0 & (k,j) \in [2^{2m-1}, 2^{2m}-1] \times [2^{2n-1}, 2^{2n}-1] \\ 1 & \textit{otherwise} . \end{cases}$$

For all $(x,y) \in D$ and let $\alpha(n) = 2^{2n-1}$, $\beta(n) = 2^{2n}-1$, $\alpha(m) = 2^{2m-1}$ and $\beta(m) = 2^{2m}-1$, then

$$P - lim_{n,m} \frac{|\{(k,j) \in [2^{2m-1}, 2^{2m}-1] \times [2^{2n-1}, 2^{2n}-1] : \| g_{k,j} - g \|_{C(D)} \geq \varepsilon\}|}{(2^{2m-1})(2^{2n-1})} = 0,$$

where g(x, y)=0 for all $(x,y) \in D$. Then for $m,n \to \infty$, $st_{\alpha\beta} - g_{m,n} \rightrightarrows g$, but since

$$\delta(\{1 \leq k \leq n, 1 \leq j \leq m: \| g_{k,j} - g \|_{C(D)} \geq \varepsilon\})$$

does not exist, $g_{k,j}$ is not uniformly convergent to g in statistical and ordinary sense. In the following, we are going to define weighted $\alpha\beta$-statistical pointwise convergence of order (γ,η), weighted $\alpha\beta$-equistatistical convergence of order (γ,η) and weighted $\alpha\beta$-statistical uniform convergence of order (γ,η).

Definition (2.9) A double sequences of bivariate functions $\{f_{m,n}\}$ on $X^2 \subseteq \mathbb{R} \times \mathbb{R}$ is said to be weighted $\alpha\beta$-statistical pointwise convergent of order (γ,η) to f if for each positive ε and for each $(x,y) \in X^2$

$$P - lim_{n,m} \frac{|\{(k,j) \in [C_m, D_m] \times [A_n, B_n] : p_k q_j |f_{k,j}(x,y) - f(x,y)| \geq \varepsilon\}|}{(B_n - A_n + 1)^{\eta} (D_m - C_m + 1)^{\gamma}} = 0.$$

Then it is shown by $w - st_{\alpha\beta}^{\gamma\eta} - f_{m,n} \to f$.

Definition (2.10) A double sequences of bivariate functions $\{f_{m,n}\}$ on $X^2 \subseteq \mathbb{R} \times \mathbb{R}$ is said to be weighted $\alpha\beta$-equistatistical convergent of order (γ,η) to f for each positive ε and each $(x,y) \in X^2$ the sequence of real valued functions

$$p_{m,n,\varepsilon,\gamma,\eta}(x,y) = \frac{|\{(k,j) \in [C_m, D_m] \times [A_n, B_n] : p_k q_j |f_{k,j}(x,y) - f(x,y)| \geq \varepsilon\}|}{(B_n - A_n + 1)^{\eta} (D_m - C_m + 1)^{\gamma}},$$

be such that the $P - lim_{m,n} \| p_{m,n,\varepsilon,\gamma,\eta}(.,.) \|_{C(X^2)} = 0$. It means that $p_{m,n,\varepsilon,\gamma,\eta}(x,y)$ converges uniformly to zero, where $\| f \|_{C(X^2)} = sup_{(x,y) \in X^2} |f(x,y)|$. Then it is shown by $w - st_{\alpha\beta}^{\gamma\eta} - f_{m,n} \twoheadrightarrow f$.

Definition (2.11) A double sequences of bivariate functions $\{f_{m,n}\}$ on $X^2 \subseteq \mathbb{R} \times \mathbb{R}$ is said to be weighted $\alpha\beta$-statistical uniform convergent of order (γ,η) to f for each positive ε,

$$P - lim_{m,n} \frac{|\{(k,j) \in [C_m, D_m] \times [A_n, B_n]: p_k q_j \|f_{k,j} - f\|_{C(X^2)} \geq \varepsilon\}|}{(B_n - A_n + 1)^{\eta} (D_m - C_m + 1)^{\gamma}} = 0.$$

Then it is shown by $w - st_{\alpha\beta}^{\gamma\eta} - f_{m,n} \rightrightarrows f$.

Example (2.12) Lets consider the sequence of continuous functions

$$h_{m,n}(x,y) = \begin{cases} 16n^2m^2(n+1)^2(m+1)^2\left(x - \frac{1}{n}\right)\left(y - \frac{1}{m}\right)\left(x - \frac{1}{n+1}\right)\left(y - \frac{1}{m+1}\right) & (k,j) \in [2^{2m-1}, 2^{2m} - 1] \times [2^{2n-1}, 2^{2n} - 1] \\ 0 & \textit{otherwise,} \end{cases}$$

where

$$(x,y) \in \left(\frac{1}{n+1}, \frac{1}{n}\right] \times \left(\frac{1}{m+1}, \frac{1}{m}\right],$$

and let h(x, y)=0, $p_k = k$, $q_j = j$ and $(\alpha, \beta) \in \Lambda$, such that $\alpha(n) = \alpha(m) = 1$ and $\beta(n), \beta(m) \in \mathbb{N}$ for all n, m. Thus $A_n = C_m = 1$ and $B_n = \beta(n)(\beta(n) + 1)/2$, $D_m = \beta(m)(\beta(m) + 1)/2$. For arbitrary $\varepsilon > 0$ and for any $0 < \gamma, \eta \leq 1$ the following is held,

$$p_{r_1, r_2, \varepsilon, \gamma, \eta}(x,y) = \frac{\left|\{(k,j) \in [1, D_m] \times [1, B_n]: p_k q_j \left|h_{k,j}(x,y) - h(x,y)\right| \geq \varepsilon\}\right|}{(D_m)^{\gamma} (B_n)^{\eta}} \leq \frac{1}{(D_m)^{\gamma} (B_n)^{\eta}} \to 0,$$

as $m, n \to \infty$ uniformly in (x, y) which gives that $w - st_{\alpha\beta}^{\gamma} - h_{m,n} \twoheadrightarrow h$, but $w - st_{\alpha\beta}^{\gamma} - h_{m,n} \rightrightarrows h$ does not hold since for any $n, m \in \mathbb{N}$,

$$\sup_{(x,y) \in [0,\infty) \times [0,\infty)} \left|h_{m,n}(x,y)\right| = 1.$$

Example (2.13) Let the bivariate functions $f_{m,n}: D = [0,\infty) \times [0,\infty) \to [0,1]$ be such that,

$$f_{m,n}(x,y) = (\frac{x}{1+x})^m (\frac{y}{1+y})^n,$$

and let $p_k = 2k$ and $q_j = 2j$, so for $(\alpha, \beta) \in \Lambda$ if we take $\alpha(n) = \alpha(m) = 1$ and $\beta(n), \beta(m) \in \mathbb{N}$, then for all n, m we have $A_n = C_m = 1$, $B_n = \beta(n)(\beta(n) + 1)$ and $D_m = \beta(m)(\beta(m) + 1)$. On the other hand since f(x, y)=0 is the pointwise limit of the sequence $f_{m,n}(x,y)$ in the ordinary sense, it is clear that $w - f_{m,n} \to f(\alpha\beta - stat)$. Choosing ε=1/4 and for all $k \in [1, D_m]$,

$$(x,y) \in (\frac{1}{\beta(m)\sqrt{2}} - 1, \infty) \times (\frac{1}{\beta(n)\sqrt{2}} - 1, \infty)$$

we have,

$$f_{m,n}(x,y) = (\frac{x}{1+x})^m(\frac{y}{1+y})^n \geq (\frac{1}{\beta(m)\sqrt{2}})^k(\frac{1}{\beta(n)\sqrt{2}})^j \geq (\frac{1}{\beta(m)\sqrt{2}})^{\beta(m)}(\frac{1}{\beta(n)\sqrt{2}})^{\beta(n)} \geq \frac{1}{4},$$

which means that $w - st_{\alpha\beta}^{\gamma,\eta} - f_{m,n} \twoheadrightarrow f$ does not hold for all $0 < \gamma, \eta \leq 1$.

3. Korovkin type approximation

Reference [28] initiated the Korovkin type approximation theory. Many different researchers in [2,3,4,5,7,8,9,11,15,16,18,19,22,25,26,31,32,33,36,38,39] have used this theory by means of statistical convergence, statistical uniformly convergence, equistatistical convergence, $\alpha\beta$ -statistical convergence, statistical C_1 summability and etc.

In this paper, we consider the space D_{ω_2} of real-valued functions, defined on E and satisfying

$$|f(u,v) - f(x,y)| \leq \omega_2\left(f; \left|\frac{u}{1+u} - \frac{x}{1+x}\right|, \left|\frac{v}{1+v} - \frac{y}{1+y}\right|\right),$$

where ω_2 is defined as a non-negative and increasing function on $E = [0,\infty) \times [0,\infty)$, such that $lim_{\delta_1,\delta_1 \to \infty}$ $\omega_2(f;\delta_1,\delta_2)$=0 and we have the following;

1) $\omega_2(f;\delta_1+\delta_2,\delta) \leq \omega_2(f;\delta_1,\delta) + \omega_2(f;\delta_2,\delta)$.

2) $\omega_2(f;\delta,\delta_1+\delta_2) \leq \omega_2(f;\delta,\delta_1)$ $+\omega_2(f;\delta,\delta_2)$.

Theorem (3.1) Let $L_{m,n}: D_{w_2} \to C_B(E)$ be a sequence of positive linear operators, if $0 < \gamma, \eta \leq 1$ and let $(\alpha,\beta) \in \Lambda$, then for all $f \in D_{w_2}$,

$$st_{\alpha\beta}^{\gamma\eta} - L_{m,n}(f;x,y) \twoheadrightarrow f(x,y) \qquad (3.1)$$

if and only if

$$st_{\alpha\beta}^{\gamma\eta} - L_{m,n}(\varphi_i;x,y) \twoheadrightarrow \varphi_i(x,y) \qquad (3.2)$$

for i=0,1,2,3 where $\varphi_0(u,v) = 1$, $\varphi_1(u,v) = \frac{u}{1+u}$, $\varphi_2(u,v) = \frac{v}{1+v}$, $\varphi_3(u,v) = \varphi_1^2(u,v) + \varphi_2^2(u,v)$.

Proof. Suppose that (3.2) holds, now take a fixed point $(x,y) \in E$ as an arbitrary point and let $f \in D_{w_2}$ so there exits δ_1, δ_2 such that $|f(u,v) - f(x,y)| < \varepsilon$, holds for all $(u,v) \in D$ satisfying,

$|\frac{u}{1+u} - \frac{x}{1+x}| < \delta_1$ and $|\frac{v}{1+v} - \frac{y}{1+y}| < \delta_2$.

Let E_{δ_1,δ_2} be the set of all $(u,v) \in E$ such that

$|\frac{u}{1+u}-\frac{x}{1+x}|<\delta_1$ and $|\frac{v}{1+v}-\frac{y}{1+y}|<\delta_2$

so clearly we have the following,

$$|f(u,v)-f(x,y)|<\varepsilon+2N\chi_{E\setminus E_{\delta_1,\delta_2}}(u,v),$$

where χ_E denotes the characteristic function of the set E and $N=\| f \|_{C_B(K)}$. On the other hand,

$$\chi_{E\setminus E_{\delta_1,\delta_2}}(u,v)\le\frac{1}{\delta_1^2}(\frac{u}{1+u}-\frac{x}{1+x})^2+\frac{1}{\delta_2^2}(\frac{v}{1+v}-\frac{y}{1+y})^2.$$

Taking $\delta=min\{\delta_1,\delta_2\}$ in the last two inequalities we have,

$$|f(u,v)-f(x,y)|\le\varepsilon+\frac{2N}{\delta^2}\{(\frac{u}{1+u}-\frac{x}{1+x})^2+(\frac{v}{1+v}-\frac{y}{1+y})^2\}. \quad (3.3)$$

By linearity and positivity of the operators $L_{m,n}$, clearly we have,

$$|L_{m,n}(f;x,y)-f(x,y)|\le\varepsilon L_{m,n}(\varphi_0;x,y)+\frac{2N}{\delta^2}L_{m,n}((\frac{u}{1+u}-\frac{x}{1+x})^2;x,y)$$

$$+L_{m,n}((\frac{v}{1+v}-\frac{y}{1+y})^2;x,y)+N|L_{m,n}(\varphi_0;x,y)-\varphi_0(x,y)|.$$

Using the boundedness of f, (3.3) and normal process of Korovkin theorem we have,

$$|L_{m,n}(f;x,y)-f(x,y)\le\varepsilon+(\varepsilon+N+\frac{4N}{\delta^2})|L_{m,n}(\varphi_0;x,y)-\varphi_0(x,y)|$$

$$+\frac{4N}{\delta^2}\{|L_{m,n}(\varphi_1;x,y)-\varphi_1(x,y)|+|L_{m,n}(\varphi_2;x,y)-\varphi_2(x,y)|\}$$

$$+\frac{2N}{\delta^2}|L_{m,n}(\varphi_3;x,y)-\varphi_3(x,y)|.$$

Let $A=\varepsilon+N+\frac{4M}{\delta^2}$, then we have

$$|L_{m,n}(f;x,y)-f(x,y)|\le\varepsilon+A\sum_{i=0}^{3}|L_{m,n}(\varphi_i;x,y)-\varphi_i(x,y)|. \quad (3.4)$$

Now choose $0<\varepsilon<l$ for any given l and define the following sets:

$$T_l(x,y)=\{(k,j)\in[\alpha(m),\beta(m)]\times[\alpha(n),\beta(n)]:|L_{k,j}(f;x,y)-f(x,y)|\ge l\}$$

$$T_l^i(x,y)=\{(k,j)\in[\alpha(m),\beta(m)]\times[\alpha(n),\beta(n)]:|L_{k,j}(\varphi_i;x,y)-\varphi_i(x,y)|\ge\frac{l-\varepsilon}{4B}\}$$

for i=0,1,2,3, obviously

$$T_l(x,y) \subset \bigcup_{i=0}^{3} T_l^i(x,y). \tag{3.5}$$

On the other hand the following real functions are defined as

$$p_{m,n,l,\gamma,\eta}(x,y) = \frac{|\{(k,j)\in[\alpha(m),\beta(m)]\times[\alpha(n),\beta(n)]:|L_{k,j}(f;x,y)-f(x,y)|\geq\frac{l-\varepsilon}{4B}\}|}{(\beta(m)-\alpha(m)+1)^{\eta}(\beta(n)-\alpha(n)+1)^{\gamma}}$$

and

$$p_{m,n,l,\gamma,\eta}^{i}(x,y) = \frac{|\{(k,j)\in[\alpha(m),\beta(m)]\times[\alpha(n),\beta(n)]:|L_{k,j}(\varphi_i;x,y)-\varphi_i(x,y)|\geq\frac{l-\varepsilon}{4B}\}|}{(\beta(n)-\alpha(n)+1)^{\gamma}(\beta(m)-\alpha(m)+1)^{\eta}}$$

for i=0,1,2,3 and $0 < \gamma, \eta \leq 1$. As a consequence of (3.5) we have

$$p_{m,n,l,\gamma,\eta}(x,y) \leq \sum_{i=0}^{3} p_{m,n,l,\gamma,\eta}^{i}(x,y) \tag{3.6}$$

for all $(x,y) \in E$. Taking supremum and limit on both sides of (3.6) we get,

$$lim_{n,m} \parallel p_{m,n,l,\gamma,\eta}(.,.) \parallel_{C_B(D)} \leq lim_{n,m} \sum_{i=0}^{3} \parallel p_{m,n,l,\gamma,\eta}^{i}(.,.) \parallel_{C_B(D)},$$

as $m,n \to \infty$, and using (3.2) we obtain (3.1) which completes the proof of (3.2) ⇒ (3.1). The inverse side is clear.

Theorem (3.2) Take $L_{m,n}: D_{w_2} \to C_B(E)$, $0 < \gamma, \eta \leq 1$ and let $(\alpha, \beta) \in \Lambda$. Thus for any function f in D_{ω_2}.

$$st_{\alpha\beta}^{\gamma\eta} - L_{m,n}(f;x,y) \rightrightarrows f(x,y) \tag{3.7}$$

if and only if

$$st_{\alpha\beta}^{\gamma\eta} - L_{m,n}(\varphi_i;x,y) \rightrightarrows \varphi_i(x,y) \tag{3.8}$$

for i=0,1,2,3, where $\varphi_0(u,v) = 1$, $\varphi_1(u,v) = \frac{u}{1+u}$, $\varphi_2(u,v) = \frac{v}{1+v}$,$\varphi_3(u,v) = \varphi_1(u,v)^2 + \varphi_2(u,v)^2$.

Proof. Applying the same steps of the proof of Theorem 3.1 and taking supremum over $(x,y) \in E$ from (3.6) we get the following inequality,

$$\parallel L_{m,n}f - f \parallel \leq A\{\parallel L_{m,n}\varphi_0 - \varphi_0 \parallel + \parallel L_{m,n}\varphi_1 - \varphi_1 \parallel + \parallel L_{m,n}\varphi_2 - \varphi_2 \parallel + \parallel L_{m,n}\varphi_3 - \varphi_3 \parallel\} + \varepsilon,$$

where $A = \varepsilon + N + \frac{4N}{\delta^2}$. Now for a given a > 0, choose $0 < \varepsilon < a$ with the following sets

$$H^{\alpha,\beta} := \{(k,j) \in [\alpha(m),\beta(m)] \times [\alpha(n),\beta(n)]: \parallel L_{k,j}(f;x,y) - f(x,y) \parallel_{C_B(D)} \geq a\}$$

$H_i^{\alpha,\beta} := \left\{(k,j) \in [\alpha(m),\beta(m)] \times [\alpha(n),\beta(n)] : \| L_{k,j}(\varphi_i;x,y) - \varphi_i(x,y) \|_{C_B(E)} \geq \frac{a-\varepsilon}{4A}\right\}$, i=1,2,3.

Therefor we have

$$H^{\alpha,\beta} \subset \bigcup_{i=0}^{3} H_i^{\alpha,\beta}.$$

This implies that,

$$\delta^{\alpha,\beta}(H^{\alpha,\beta},\gamma,\eta) \leq \sum_{i=0}^{3} \delta^{\alpha,\beta}(H_i^{\alpha,\beta},\gamma,\eta)$$

using (3.8), completes the proof. The implication (3.7)⇒ (3.8) is clear.

Considering the following operators;

$$B_{m,n}(f,x,y) = \frac{1}{(1+x)^m(1+y)^n} \sum_{k=0}^{n}\sum_{l=0}^{m} f\left(\frac{k}{m-k+1},\frac{l}{n-l+1}\right)\binom{m}{k}\binom{n}{l} x^k y^l,$$

where $E = [0,\infty) \times [0,\infty)$, $f \in D_{w_2}$, $(x,y) \in E$ and $n \in \mathbb{N}$.

Using $B_{m,n}(f,x,y)$ we can introduce the following positive linear operators;

$$T_{m,n}(f;x,y) = (1 + h_{m,n}(x,y))B_{m,n}(f;x,y),$$

where $h_{m,n}(x,y)$ is the double sequences of bivariate functions, given in the Example 2.12. The following are easily seen.

$$T_{m,n}(\varphi_0;x,y) = 1 + h_{m,n}(x,y)$$

$$T_{m,n}(\varphi_1;x,y) = (1 + h_{m,n}(x,y))\left(\frac{m}{1+m}\right)\left(\frac{x}{1+x}\right)$$

$$T_{m,n}(\varphi_2;x,y) = (1 + h_{m,n}(x,y))\left(\frac{n}{1+n}\right)\left(\frac{y}{1+y}\right)$$

$$T_{m,n}(\varphi_3;x,y) = (1 + h_{m,n}(x,y))\Big\{\frac{m(m-1)}{(m+1)^2}\frac{x^2}{(1+x)^2} + \frac{m}{(m+1)^2}\frac{x}{1+x}$$

$$+\frac{n(n-1)}{(n+1)^2}\frac{y^2}{(1+y)^2} + \frac{n}{(n+1)^2}\frac{y}{1+y}\Big\}.$$

$T_{m,n}$ satisfies in the condition of (3.2) and as $h_{m,n}$ is weighted $\alpha\beta$-equistatistical convergent to zero of order (γ,η) so by Theorem (3.1) we have,

$$st_{\alpha\beta}^{\gamma\eta} - T_{m,n}(f;x,y) \twoheadrightarrow f(x,y). \tag{3.9}$$

But, $st_{\alpha\beta}^{\gamma\eta} - h_{m,n} \not\rightrightarrows 0$ and the condition (3.8) does not hold, therefore

$st_{\alpha\beta}^{\gamma\eta} - T_{m,n}(f; x, y) \rightrightarrows f(x, y)$,

does not hold. It means, $\alpha\beta$-equistatistical convergence of order (γ, η) can not be changed by $\alpha\beta$-statistical uniform convergence of order (γ, η), in (3.9).

Theorem (3.3) Let $L_{m,n}: D_{w_2} \to C_B(E)$, $0 < \gamma, \eta \leq 1$, $(\alpha, \beta) \in \Lambda$ and let p_n and q_m be sequences satisfying (2.1) and (2.2). Thus for all $f \in D_{w_2}$,

$$w - st_{\alpha\beta}^{\gamma\eta} - L_{m,n}(f; x, y) \twoheadrightarrow f(x, y) \tag{3.10}$$

if and only if

$$w - st_{\alpha\beta}^{\gamma\eta} - L_{m,n}(\varphi_i; x, y) \twoheadrightarrow \varphi_i(x, y) \tag{3.11}$$

for i=0,1,2,3 where $\varphi_0(u, v) = 1$, $\varphi_1(u, v) = \frac{u}{1+u}$, $\varphi_2(u, v) = \frac{v}{1+v}$, $\varphi_3(u, v) = \varphi_1(u, v)^2 + \varphi_2(u, v)^2$.

Proof. Using (3.4) we have the following equation,

$$|L_{m,n}(f; x, y) - f(x, y)| \leq \varepsilon + A \sum_{i=0}^{3} |L_{m,n}(\varphi_i; x, y) - \varphi_i(x, y)|,$$

where $A = \varepsilon + N + \frac{4M}{\delta^2}$. Now if $0 < \varepsilon < s$ for any arbitrary s, then we can define the following sets,

$$R_s(x, y) = \{(k, j) \in [C_m, D_m] \times [A_n, B_n]: p_k q_j |L_{k,j}(f; x, y) - f(x, y)| \geq s\}$$

$$R_s^i(x, y) = \{(k, j) \in (k, j) \in [C_m, D_m] \times [A_n, B_n]: p_k q_j |L_{k,j}(\varphi_i; x, y) - \varphi_i(x, y)| \geq \tfrac{s-\varepsilon}{4B}\}$$

for i=0,1,2,3. It is obvious that

$$R_s(x, y) \subset \bigcup_{i=0}^{3} R_s^i(x, y). \tag{3.12}$$

Now define the following real valued functions:

$$h_{m,n,,s,\gamma,\eta}(x, y) = \frac{|\{(k, j) \in [C_m, D_m] \times [A_n, B_n]: p_k q_j |L_{k,j}(f; x, y) - f(x, y)| \geq \frac{s-\varepsilon}{4B}\}|}{(D_m - C_m + 1)^\gamma (B_n - A_n + 1)^\eta}$$

and

$$h^{i}_{m,n,s,\gamma,\eta}(x,y) = \frac{|\{(k,j) \in [C_m, D_m] \times [A_n, B_n]: p_k q_j |L_k(\varphi_i; x, y) - \varphi_i(x,y)| \geq \frac{s-\varepsilon}{4B}\}|}{(D_m - C_m + 1)^{\gamma}(B_n - A_n + 1)^{\eta}}$$

for i=0,1,2,3 and $0 < \gamma, \eta \leq 1$. Then as a consequence of (3.12) we have

$$h_{m,n,s,\gamma,\eta}(x,y) \leq \sum_{i=0}^{3} h^{i}_{m,n,s,\gamma,\eta}(x,y) \qquad (3.13)$$

for all $(x,y) \in D$. Applying supremum and limit on both sides of (3.13) we get,

$$lim_{m,n} \parallel h_{m,n,s,\gamma,\eta}(.) \parallel_{C_B(D)} \leq lim_{m,n} \sum_{i=0}^{3} \parallel h^{i}_{m,n,s,\gamma,\eta}(.) \parallel_{C_B(D)}.$$

as $m, n \to \infty$ and using (3.11) we obtain (3.10) which completes the proof of (3.11) ⇒ (3.10). The inverse implication is clear.

Theorem (3.4) Let $L_{m,n}: D_{w_2} \to C_B(E)$, $0 < \gamma, \eta \leq 1$, $(\alpha, \beta) \in \Lambda$ and let p_n and q_m be sequences satisfying (2.1) and (2.2). Therefore for all $f \in D_{w_2}$,

$$w - st^{\gamma\eta}_{\alpha\beta} - L_{m,n}(f; x, y) \rightrightarrows f(x,y) \qquad (3.14)$$

if and only if

$$w - st^{\gamma\eta}_{\alpha\beta} - L_{m,n}(\varphi_i; x, y) \rightrightarrows \varphi_i(x,y) \qquad (3.15)$$

for i=0,1,2,3 where $\varphi_0(u,v) = 1$, $\varphi_1(u,v) = \frac{u}{1+u}$, $\varphi_2(u,v) = \frac{v}{1+v}$, $\varphi_3(u,v) = \varphi_1(u,v)^2 + \varphi_2(u,v)^2$.

Proof. Applying supremum to both sides of (3.13), then we have the following,

$$\parallel L_{m,n}f - f \parallel \leq A\{\parallel L_{m,n}\varphi_0 - \varphi_0 \parallel + \parallel L_{m,n}\varphi_1 - \varphi_1 \parallel + \parallel L_{m,n}\varphi_2 - \varphi_2 \parallel + \parallel L_{m,n}\varphi_3 - \varphi_3 \parallel\} + \varepsilon,$$

where $A = \varepsilon + M + \frac{4M}{\delta^2}$. Now for a given t > 0, choose $0 < \varepsilon < t$ and define the following sets:

$$G^{\alpha,\beta} := \left\{(k,j) \in [C_m, D_m] \times [A_n, B_n]: p_k q_j \parallel L_{k,j}(f; x, y) - f(x,y) \parallel_{C_B(D)} \geq t\right\}$$

$$G^{\alpha,\beta}_i := \left\{(k,j) \in [C_m, D_m] \times [A_n, B_n]: p_k q_j \parallel L_{k,j}(\varphi_i; x, y) - \varphi_i(x,y) \parallel_{C_B(D)} \geq \frac{t-\varepsilon}{4B}\right\} i = 0,1,2,3.$$

So we have

$$G^{\alpha,\beta} \subset \bigcup_{i=0}^{3} G^{\alpha,\beta}_i.$$

This implies that,

$$\delta^{\alpha,\beta}(G^{\alpha,\beta},\gamma,\eta) \leq \sum_{i=0}^{3} \delta^{\alpha,\beta}(G_i^{\alpha,\beta},\gamma,\eta),$$

which completes the proof of (3.15) ⇒ (3.14). The inverse implication is obvious. Now lets define $T^*_{m,n}(f;x,y)$ as a positive linear operator,

$$T^*_{m,n}(f;x,y) = (1 + h_{m,n}(x,y))B_{m,n}(f;x,y),$$

where $h_{m,n}(x,y)$ is the double sequence of functions of two variables, considered in Example 2.13. If $p_n = n$, $q_m = m$, $\alpha(n) = \alpha(m) = 1$ and $\beta(n), \beta(m) \in \mathbb{N}$ for all n, m, then for any $0 < \gamma, \eta \leq 1$, the double sequence of positive linear operators $T^*_{m,n}$ satisfy the conditions (3.11) (see Example 2.13). Hence by Theorem 3.3, we have

$$w - st_{\alpha\beta}^{\gamma\eta} - T^*_{m,n}(f;x,y) \twoheadrightarrow f(x,y),$$

but since $T^*_{m,n}$ does not satisfy conditions (3.16)

$$w - st_{\alpha\beta}^{\gamma\eta} - T^*_{m,n}(f;x,y) \not\rightrightarrows f(x,y).$$

4. Rates of weighted $\alpha\beta$-equistatistical convergence of order $(\boldsymbol{\gamma},\boldsymbol{\eta})$

In this part, using the modulus of continuity, the rates of weighted $\alpha\beta$-equistatistical convergence of order (γ,η) of $L_{m,n}$, as a positive linear operators defined on D_{w_2} is studied .

Definition (4.1) Considering $a_{m,n}$ as a double sequence which is non-decreasing. A double sequence of bivariate function $f_{m,n}$ is called weighted $\alpha\beta$-equistatistical convergent of order (γ,η) to function f with the rate of $a_{m,n}$ for every $\varepsilon > 0$ we have,

$$\frac{|\{(k,j) \in [C_{r_1}, D_{r_1}] \times [A_{r_2}, B_{r_2}]: p_k q_j |f_{k,j}(x,y) - f(x,y)| \geq \varepsilon\}|}{(D_{r_1} - C_{r_1} + 1)^{\gamma}(B_{r_2} - A_{r_2} + 1)^{\eta} a_{r_1,r_2}} \to 0$$

uniformly, with respect to $(x,y) \in D$, where p_k and q_j are sequences satisfying (2.1) and (2.2) in this case we show it by $w - (f_{m,n} - f) = o(a_{m,n},\gamma,\eta)(\alpha\beta - equistat)$.

Lemma (4.2) Take two double sequences of functions $f_{m,n}$ and $g_{m,n}$ in H_{w_2}, in the way that

$$w - (f_{m,n} - f) = o(a_{m,n},\gamma,\eta)(\alpha\beta - equistat)$$

and

$w-(g_{m,n}-g)=o(b_{m,n},\gamma,\eta)(\alpha\beta-equistat)$

thus the following are true,

1) $f_{m,n}+g_{m,n}$ is weighted $\alpha\beta$ -equistatistical convergent of order (γ,η) to f+g with the rate $(max\{a_{m,n},b_{m,n}\})$.

2) $(f_{m,n})(g_{m,n})$ is weighted $\alpha\beta$-equistatistical convergent of order (γ,η) to fg with the rate $a_{m,n}b_{m,n}$.

3) For any scalar number M, $M(f_{m,n})$ is weighted $\alpha\beta$-equistatistical convergent of order (γ,η) to M(f) with the rate $a_{m,n}$.

4) $w-(\sqrt{f_{m,n}-f})=o(a_{m,n},\gamma,\eta)(\alpha\beta-equistat)$.

Proof. 1) Assume that $(f_{m,n})$ is weighted $\alpha\beta$-equistatistical convergent of order (γ,η) to f with the rate of $a_{m,n}$ and $(g_{m,n})$ is weighted $\alpha\beta$-equistatistical convergent of order (γ,η) to g with the rate of $b_{m,n}$ on D. For any $\varepsilon>0$ and $(\alpha,\beta)\in\Lambda$ consider the following sets,

$$V_{m,n,\alpha,\beta}(x,y,\varepsilon)=|\{(k,j)\in[C_m,D_m]\times[A_n,B_n]:p_kq_j|(f_{k,j}+g_{k,j})(x,y)-(f+g)(x,y)|\geq\varepsilon\}|.$$

$$V^1_{m,n,\alpha,\beta}(x,y,\varepsilon)=|\{(k,j)\in[C_m,D_m]\times[A_n,B_n]:p_kq_j|f_{k,j}(x,y)-f(x,y)|\geq\frac{\varepsilon}{2}\}|.$$

$$V^2_{m,n,\alpha,\beta}(x,y,\varepsilon)=|\{(k,j)\in[C_m,D_m]\times[A_n,B_n]:p_kq_j|g_{k,j}(x,y)-g(x,y)|\geq\frac{\varepsilon}{2}\}|.$$

It is obvious that,

$$\frac{V_{m,n,\alpha,\beta}(x,y,\varepsilon)}{(D_m-C_m+1)^\gamma(B_n-A_n+1)^\eta c_{m,n}}\leq\frac{V^1_{m,n,\alpha,\beta}(x,y,\varepsilon)}{(D_m-C_m+1)^\gamma(B_n-A_n+1)^\eta a_{m,n}}$$

$$+\frac{V^2_{m,n,\alpha,\beta}(x,y,\varepsilon)}{(D_m-C_m+1)^\gamma(B_n-A_n+1)^\eta b_{m,n}}$$

$c_{m,n}=max\{a_{m,n},b_{m,n}\}$. If we apply limit to both sides of above inequity as $m,n\to\infty$ and using the hypotheses of Lemma (4.2), the proof of 1) is completed. The proof (2)-(4) can be obtain in the same way.

Theorem (4.3) Take the positive linear operator $L_{m,n}:D_{w_2}\to C(E)$ and assume the following properties are true,

i) $L_{m,n}(f_0;x,y)$ is weighted $\alpha\beta$-equistatistical convergent of order (γ,η) to f_0 with the rate of $a_{m,n}$.

ii) $\omega(f,\delta_{m,n})=o(b_{m,n},\gamma,\eta)(\alpha\beta-equistat)$ where $\delta_{m,n}=\sqrt{L_{m,n}(\phi^2;x,y)}$ with $(\phi^2;x,y)=(u-x)^2+(v-y)^2$.

Then, for all $f \in D_{w_2}$ we have, $L_{m,n}(f_0; x, y)$ is weighted $\alpha\beta$-equistatistical convergent of order (γ, η) to f with the rate of $c_{m,n}$ where $c_{m,n} = max\{a_{m,n}, b_{m,n}\}$.

Proof. Let $f \in D_{w_2}$ and let (x, y) be a fixed point of E thus it is well known that,

$$|L_{m,n}(f; x, y) - f(x, y)| \leq \| f(x, y) \|_{D_{w_2}} |L_{m,n}(f_0; x, y) - f_0(x, y)| + 2\omega(f, \delta_{m,n})|L_{m,n}(f_0; x, y) - f_0(x, y)|$$

$$+\omega(f, \delta_{m,n})\sqrt{|L_{m,n}(f_0; x, y) - f_0(x, y)|}.$$

Using the hypothesis, and Lemma 4.2, in the above inequality, completes the proof.

5. Conclusion

We use a correct modification to introduce weighted $\alpha\beta$-equistatistical convergence of order (γ, η) for double sequences of functions and also we study Korovkin type approximation theorems via weighted $\alpha\beta$-equistatistical convergence and $\alpha\beta$-statistical uniform convergence of order (γ, η) for double sequences of functions of two variables on $E = [0, \infty) \times [0, \infty)$. Approximation results are illustrated on some examples of positive linear operators to show that our definition works. We also bring some examples to show the differences between weighted $\alpha\beta$-equistatistical convergence, weighted $\alpha\beta$-statistical pointwise convergence and weighted $\alpha\beta$-statistical uniform convergence**.** Furthermore the rate of weighted $\alpha\beta$-equistatistical convergence of order (γ, η) is studied. One can consider the definition $\alpha\beta$-equistatistical convergence and weighted $\alpha\beta$-equistatistical convergence and uniform $\alpha\beta$-statistical convergence for blending type Bernstein operators.

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