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The Non-Negative P_0–Matrix Completion Problem for 5×5 Matrices Specifying Cyclic Diagraphs with 5 Vertices and 4 Arcs

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Abstract

The non-negative P0-matrix completion is considered for 5×5 matrices specifying digraphs with p=5 and q=4.The research determines which of the digraphs with p=5 and q=4 and specifying 5×5 partial matrices have non-negative P0-completion. Considering the 5×5 matrices with q=4, all the sixty one (61) non-isomorphic digraphs shall be constructed. All the partial non-negative P0-matrices will be extracted from each digraph. To establish if the pattern has non-negative P0-completion or not, zero completion will be performed on each of the partial matrix extracted. The study establishes that all acyclic digraphs of an 5×5 matrix with q=4 have non-negative P0-completion. The matrix completion problem is to find the values of an n x m matrix M, given a sparse and incomplete set of observations. Possible areas of applications include Seismic data reconstruction to recover missing traces when data is sparse and incomplete ,say due to malfunctioned measuring instruments, biased or corrupted traces, ground barriers, or due to financial limitation to access complete data. Others include incomplete market surveys (eg movie ratings to complete missing data so as to recommend appropriately to viewers), weather forecasting from historical data recordings as well as future predictions from computer simulations, reconstruction of images in computer; and finding the positions of sensors in Global Positioning from distances available in a local network.

Keywords: graph; subgraph; directed graph; cyclic digraph; acyclic digraph; complete digraph; path; cycle; zero completion; isomorphic; partial matrix; sub-matrix; principal minor; P_0-matrix; non-negative P_0-matrix.

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1. Introduction

All 3×3 matrices have non-negative P_0 completion Hogben [1]. A pattern for 4×4 matrices that includes all diagonal positions has non-negative P_0 completion if and only if its digraph is complete when it has a 4-cycle Choi and his colleagues [2]. They also showed that any positionally symmetric pattern that includes all diagonal positions and whose graph is an *n*-cycle has non-negative P_0 completion if and only if $n \neq 4$. All digraphs for $p = 5$, $q = 3$ specifying 5×5 partial matrices which are either cycles or acyclic have non-negative P_0 completion Munyiri and his colleagues [3]. It is possible to work out the number of digraphs (with n points and q lines). Harary and his colleagues [4], developed a technique for working out the number of digraphs (with n points and q lines) and established the following results.

Table 1

No. of edges(q)	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Greater or equal to 11**
No. of digraphs	**1**	**1**	**5**	**16**	**61**	**154**	**379**	**707**	**1155**	**1490**	**1670**	**3969**

Therefore, there are sixty one (61) non-isomorphic digraphs with 5 points and 4 arcs. Out of this, 5 are cyclic while 56 are acyclic. All the 5 cyclic non-isomorphic digraphs of 5×5 matrices with $p = 5$, $q = 4$ are discussed. All the digraphs are assumed to include all diagonal positions. Each digraph shall be considered as a separate case.

2. Mathematical Analysis

Consider the digraph shown below.

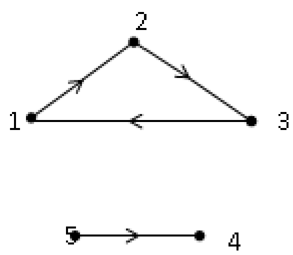

Figure 1

To obtain a partial matrix from the digraph an arrow pointing from one vertex to another, say 1 to 2 represents a specified entry (a_{ij}), while for an unspecified entry there is no arrow pointing from one vertex to the other and is denoted by x_{ij}. Therefore the partial matrix representing the above digraph is as below:

$$\begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ a_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & a_{54} & d_{55} \end{bmatrix}$$

The other digraphs and their corresponding partial non-negative P_0- matrices can be constructed and extracted, respectively, in a similar manner.

By definition of partial non-negative P_0 - matrix, $d_{11} \geq 0, d_{22} \geq 0, d_{33} \geq 0, d_{44} \geq 0, d_{55} \geq 0$. Next, we compute the principal minors.

$DetB(1,2) = d_{11}d_{22} - a_{12}x_{21}$. Setting the unspecified entry to zero, i.e., $x_{21} = 0,$ then $Det\,\mathrm{B}(1,2) = d_{11}d_{22} \geq 0$. Similarly, $\det B(1,3)$, $\det B(1,4)$, $Det\,\mathrm{B}(1,5)$, $Det\,\mathrm{B}(2,3)$, $Det\,\mathrm{B}(2,4)$, $Det\,\mathrm{B}(2,5)$, $Det\,\mathrm{B}(3,5)$, $Det\,\mathrm{B}(4,5) \geq 0$.

Substitute zero for the unknown entries; these are

$$x_{13} = x_{14} = x_{15} = x_{21} = x_{24} = x_{25} = x_{32} = x_{34} = x_{35} = x_{41} = x_{42} = x_{43} = x_{45} = x_{51} = x_{52} = x_{53} = 0,$$

We get $DetB(1,2,3) = d_{11}d_{22}d_{33} \geq 0$. Similarly,

$$Det\,\mathrm{B}(1,2,4), Det\,\mathrm{B}(1,2,5), Det\,\mathrm{B}(1,3,4), Det\,\mathrm{B}(1,3,5), Det\,\mathrm{B}(1,4,5), Det\,\mathrm{B}(2,3,4), Det\,\mathrm{B}(2,3,5), Det\,\mathrm{B}(2,4,5), Det\,\mathrm{B}(3,4,5) \geq 0$$

$$\begin{aligned} Det\,\mathrm{B}(1,2,3,4) &= d_{11}\left[d_{22}d_{33}d_{44} + x_{24}x_{32}x_{43} + a_{23}x_{34}x_{42} - x_{24}d_{33}x_{42} - d_{22}x_{34}x_{43} - a_{23}x_{32}\,d_{44}\right] \\ &-a_{12}\left[x_{21}d_{33}d_{44} + x_{24}a_{31}x_{43} + a_{23}x_{34}x_{41} - x_{24}d_{33}x_{41} - x_{21}x_{34}x_{43} - a_{23}x_{31}d_{44}\right] \\ &+x_{13}\left[x_{21}x_{32}d_{44} + x_{24}a_{31}x_{42} + d_{22}x_{34}x_{41} - x_{24}x_{32}x_{41} - x_{21}x_{34}x_{42} - d_{22}a_{31}d_{44}\right] \\ &-x_{14}\left[x_{21}x_{32}x_{43} + a_{23}x_{31}x_{42} + d_{22}d_{33}x_{41} - a_{23}x_{32}x_{41} - x_{21}d_{33}x_{42} - d_{22}a_{31}x_{43}\right] \end{aligned}$$

Substitute zero for the unknown entries; these are

$$x_{13} = x_{14} = x_{15} = x_{21} = x_{24} = x_{25} = x_{32} = x_{34} = x_{35} = x_{41} = x_{42} = x_{43} = x_{45} = x_{51} = x_{52} = x_{53} = 0,$$

We get $Det\,\mathrm{B}(1,2,3,4) = d_{11}d_{22}d_{33}d_{44} \geq 0$. Similarly,

$Det\,\mathrm{B}(1,2,3,5), Det\,\mathrm{B}(1,2,4,5), Det\,\mathrm{B}(1,3,4,5), Det\,\mathrm{B}(2,3,4,5) \geq 0.$

$$\begin{aligned}
Det\,\mathrm{B}(1,2,3,4,5) = d_{11}\Big(& d_{22}\left[d_{33}d_{44}d_{55} + x_{35}x_{43}a_{54} + x_{34}x_{45}x_{53} - x_{35}d_{44}x_{53} - d_{33}x_{45}a_{54} - x_{34}x_{43}d_{55}\right] \\
& -a_{23}\left[x_{32}d_{44}d_{55} + x_{35}x_{42}a_{54} + x_{34}x_{45}x_{52} - x_{35}d_{44}x_{52} - x_{32}x_{45}a_{54} - x_{34}x_{42}d_{55}\right] \\
& +x_{24}\left[x_{32}x_{43}d_{55} + x_{35}x_{42}x_{53} + d_{33}x_{45}x_{52} - x_{35}x_{43}x_{52} - x_{32}x_{45}x_{53} - d_{33}x_{42}d_{55}\right] \\
& -x_{25}\left[x_{32}x_{43}a_{54} + x_{34}x_{42}x_{53} + d_{33}d_{44}x_{52} - x_{34}x_{43}x_{52} - x_{32}d_{44}x_{53} - d_{33}x_{42}a_{54}\right]\Big)
\end{aligned}$$

$$\begin{aligned}
-a_{12}\Big(& x_{21}\left[d_{33}d_{44}d_{55} + x_{35}x_{43}a_{54} + x_{34}x_{45}x_{53} - x_{35}d_{44}x_{53} - d_{33}x_{45}a_{54} - x_{34}x_{43}d_{55}\right] \\
& -a_{23}\left[a_{31}d_{44}d_{55} + x_{35}x_{41}a_{54} + x_{34}x_{45}x_{51} - x_{35}d_{44}x_{51} - x_{31}x_{45}a_{54} - x_{34}x_{41}d_{55}\right] \\
& +x_{24}\left[a_{31}x_{43}d_{55} + x_{35}x_{41}x_{53} + d_{33}x_{45}x_{51} - x_{35}x_{43}x_{51} - a_{31}x_{45}x_{53} - d_{33}x_{41}d_{55}\right] \\
& -x_{25}\left[a_{31}x_{43}a_{54} + x_{34}x_{41}x_{53} + d_{33}d_{44}x_{51} - x_{34}x_{43}x_{51} - a_{31}d_{44}x_{53} - d_{33}x_{41}a_{54}\right]\Big)
\end{aligned}$$

$$\begin{aligned}
+x_{13}\Big(& x_{21}\left[x_{32}d_{44}d_{55} + x_{35}x_{41}a_{54} + x_{34}x_{45}x_{52} - x_{35}d_{44}x_{52} - x_{32}x_{45}a_{54} - x_{34}x_{42}d_{55}\right] \\
& -d_{22}\left[a_{31}d_{44}d_{55} + x_{35}x_{41}a_{54} + x_{34}x_{45}x_{51} - x_{35}d_{44}x_{51} - x_{31}x_{45}a_{54} - x_{34}x_{41}d_{55}\right] \\
& +x_{24}\left[a_{31}x_{41}d_{55} + x_{35}x_{41}x_{52} + x_{32}x_{45}x_{51} - x_{35}x_{42}x_{51} - a_{31}x_{45}x_{52} - x_{32}x_{41}d_{55}\right] \\
& -x_{25}\left[x_{31}x_{42}a_{54} + x_{34}x_{41}x_{52} + x_{32}d_{44}x_{51} - x_{34}x_{41}x_{51} - a_{31}d_{44}x_{52} - x_{32}x_{41}a_{54}\right]\Big)
\end{aligned}$$

$$\begin{aligned}
+x_{14}\Big(& x_{21}\left[x_{32}x_{43}d_{55} + x_{35}x_{42}x_{53} + d_{33}x_{45}x_{52} - x_{35}x_{53}x_{52} - x_{32}x_{45}x_{53} - d_{33}x_{42}d_{55}\right] \\
& -d_{22}\left[a_{31}x_{43}d_{55} + x_{35}x_{41}x_{53} + d_{33}x_{45}x_{51} - x_{35}x_{43}x_{51} - a_{31}x_{45}x_{53} - d_{33}x_{41}d_{55}\right] \\
& +a_{23}\left[a_{31}x_{42}d_{55} + x_{35}x_{41}x_{52} + x_{32}x_{45}x_{51} - x_{35}x_{42}x_{51} - x_{31}x_{45}x_{52} - x_{41}x_{32}d_{55}\right] \\
& -x_{25}\left[x_{31}x_{42}x_{53} + d_{33}x_{41}x_{52} + x_{32}x_{43}x_{51} - d_{33}x_{42}x_{51} - a_{31}x_{43}x_{52} - x_{32}x_{41}x_{53}\right]\Big)
\end{aligned}$$

$$\begin{aligned}
+x_{15}\Big(& x_{21}\left[x_{32}x_{43}x_{54} + x_{34}x_{42}x_{53} + d_{33}d_{44}x_{52} - x_{34}x_{43}x_{52} - x_{32}d_{44}x_{53} - d_{33}x_{42}x_{54}\right] \\
& -d_{22}\left[x_{31}x_{43}x_{54} + x_{34}x_{41}x_{53} + d_{33}d_{44}x_{51} - x_{34}x_{43}x_{51} - x_{31}d_{44}x_{53} - d_{33}x_{41}x_{54}\right] \\
& +x_{23}\left[x_{31}x_{42}x_{54} + x_{34}x_{41}x_{52} + x_{32}d_{44}x_{51} - x_{34}x_{42}x_{51} - x_{31}d_{44}x_{52} - x_{32}x_{41}x_{54}\right] \\
& -x_{24}\left[x_{31}x_{42}x_{53} + d_{33}x_{41}x_{52} + x_{32}x_{43}x_{51} - d_{33}x_{42}x_{51} - x_{31}x_{43}x_{52} - x_{32}x_{41}x_{53}\right]\Big)
\end{aligned}$$

Substitute zero for the unknown entries; these are

$$x_{13} = x_{14} = x_{15} = x_{21} = x_{24} = x_{25} = x_{32} = x_{34} = x_{35} = x_{41} = x_{42} = x_{43} = x_{45} = x_{51} = x_{52} = x_{53} = 0,$$

to obtain $Det\,\mathrm{B}(1,2,3,4,5) = d_{11}d_{22}d_{33}d_{44}d_{55} + a_{12}a_{23}a_{31}d_{44}d_{55} \geq 0$

since all specified entries are non-negative and therefore the digraph has non-negative P_0 -completion.

A similar process is carried out for all the other three digraphs which are 3-cycles. These are as follows:

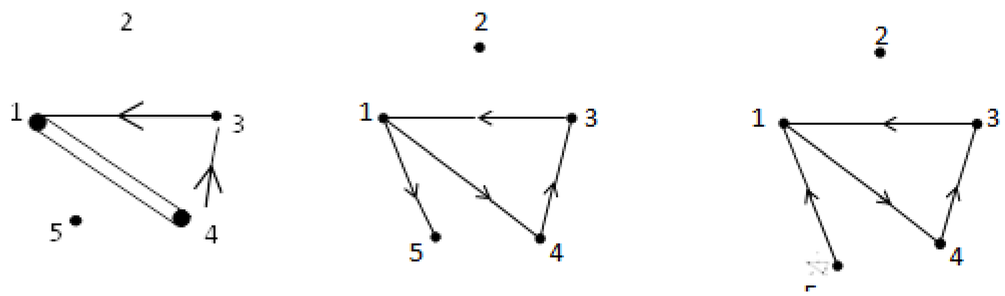

Figure 2

Now consider the digraph represented below which has a 4-cycle:

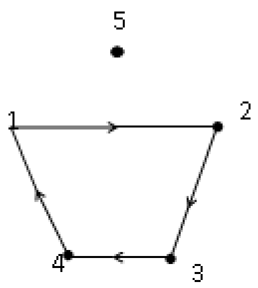

Figure 3

The partial non-negative P_0 -matrix B , arising from the digraph is then extracted from the digraph as below.

$$B = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{5} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ a_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$$

Now we show that B does not have non-negative P_0 -completion by a counter example as shown below

$$[x_{21}x_{32}d_{44} + x_{24}x_{31}x_{42} + d_{22}x_{34}x_{41} - x_{24}x_{32}x_{41} - x_{21}x_{34}x_{42} - d_{22}x_{31}d_{44}]$$

$$B = \begin{bmatrix} 0 & 1 & x_{13} & x_{14} & x_{15} \\ x_{21} & 0 & 1 & x_{24} & x_{5} \\ x_{31} & x_{32} & 0 & 1 & x_{35} \\ 1 & x_{42} & x_{43} & 0 & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & 0 \end{bmatrix}$$

$Det\,\mathrm{B}(1,2) = d_{11}d_{22} - x_{12}x_{21} = 0 - x_{21} = -x_{21} \geq 0$ which implies that $x_{21} = 0$

$Det\,\mathrm{B}(1,3) = d_{11}d_{13} - x_{13}x_{31} = 0 - x_{13}x_{31} = x_{13}x_{31} \geq 0$ which implies that $x_{13}x_{31} = 0$

Similarly, $x_{14} = x_{32} = x_{43} = 0$ and $x_{24}x_{42} = 0$. Therefore,

$$\begin{aligned} &Det\,\mathrm{B}(1,2,3,4) = d_{11}[d_{22}d_{33}d_{44} - x_{24}x_{32}x_{43} + x_{23}x_{34}x_{42} - x_{24}d_{33}x_{42} - d_{22}x_{34}x_{43} - x_{23}x_{32}d_{44}] \\ &-x_{12}[x_{21}d_{33}d_{44} + x_{24}x_{31}x_{43} + x_{23}x_{34}x_{41} - x_{24}x_{33}x_{41} - x_{21}x_{34}x_{43} - x_{23}x_{31}x_{44}] \\ &+x_{13}[x_{21}x_{32}d_{44} + x_{24}x_{31}x_{42} + d_{22}x_{34}x_{41} - x_{24}x_{32}x_{41} - x_{21}x_{34}x_{42} - d_{22}x_{31}d_{44}] \\ &-x_{14}[x_{21}x_{32}x_{43} + x_{23}x_{31}x_{42} + d_{22}d_{33}x_{41} - x_{23}x_{32}x_{41} - x_{21}d_{33}x_{42} - d_{22}x_{31}d_{43}] \\ &0-1-0-0=-1<0 \end{aligned}$$

This implies that B is not a non-negative P_0-matrix. Hence there is no non-negative P_0-completion for the digraph.

3. Results

The results show that all partial matrices for digraphs of 5× 5matrices with p=5, q=4 that are 3-cycles have non-negative P0-completion and the 4-cycle digraph does not have non-negative P0-completion.

4. Conclusion

A similar procedure was carried out on all the other remaining 3-cycle digraphs and it was found that they have non-negative P_0-completion. It was therefore established that all 3-cyclic digraphs of a 5×5 matrix with q=4 have non-negative P0-completion. However, any digraph for an n × n matrix which has the 4-cycle considered above does not have non-negative P_0-completion.

5. Recommendation

Future research can be carried out to determine appropriate applications for non-negative P0- matrix completion.

6. Validation

When q=4 and n=5 the results agrees with those of Munyiri and his colleagues [3] where q=3 and n=5. The results obtained in these research can be applied in filling partially filled data in market surveys to predict markets trends.

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