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Volatility Extraction in Information Based Asset Pricing Framework Via Non-Linear Filtering

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Abstract

This study looks at the derivation of a state space model that is applied in non-linear filtering. The model is based on the Brody, Hughson and Macrina information based asset pricing model, also known as the BHM approach or BHM model. The objective of this study is to extend the application of a filtering approach used in estimation of volatilities for the Heston model to the BHM model. The measurement and transition equations obtained in the state space model are used in the extended kalman filter to extract volatility. The option price is obtained from the BS-BHM Updated Model by incorporating information in the Black-Scholes Model. This option price is used to obtain the measurement equation while the variance process is used as the transition equation.

Keywords: Kalman filter; Extended Kalman Filter; Measurement Equation; Transition Equation; State Space Model.

1. Introduction

The author in [7] proposed a stochastic volatility model (SVM) referred to as the Heston model where the asset returns' volatility are driven by a stochastic variance process.

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Filtering has been applied in estimating the volatility from an underlying asset price. In this study, in order to apply filtering, the state space models which consist of a measurement and state transition equation are used. A similar approach to that proposed by the authors in [16] where filtering is applied to the Heston Model is followed. The approach is extended to the case of information based asset pricing in [4]. Filtering is an engineering terminology for extracting information about a signal from partial and noisy observations. Filtering can be used to estimate a dynamic system's internal states given that the system has a series of current and past noisy observations. The observation variables are observable unlike the system's states which are unobservable. The system's states conditional probability distribution can then be estimated using the filtering approach. The rationale underlying filtering is to obtain the optimal estimate of a hidden state given all the available information up to that point. The estimation is performed in two steps: The first step involves determining the prior distribution of the hidden state, denoted by the vector x_k using all the given past information. That is, the prediction of x_k at time k is given by $\hat{x}_{k|k-1}$. An assumption is made that the previous estimates $\hat{x}_{k-1}$ are known, they are used to predict the state vector $\hat{x}_{k|k-1}$. The second step involves using Bayes rule to obtain the posterior distribution by combining the prior distribution obtained in the first step with the conditional likelihood of the newest observation. This implies that the predicted states $\hat{x}_{k|k-1}$ are combined with the current observations y_k to estimate the current states $\hat{x}_{k|k}$. Given that the observations are noisy, the interest is to get the best estimate $\hat{x}_{k|k}$of x_k that minimizes the error, $x_k - \hat{x}_{k|k}$. This is done by recursion at each time step k. A Kalman filter is an example of an optimal filtering method which is applicable in the field of science, engineering and finance. In finance it can be used in hedging under partial observation, volatility estimation, optimal asset allocation, etc. It is considered easy to understand with little computational burdens. The Kalman filter is also ideal when a large volume of information must be taken into account because it is very fast. There are two basic building blocks of a Kalman Filter, the measurement equation and the transition equation. The measurement equation relates an unobserved variable to an observable variable. The transition equation is based on a model that allows the unobserved variable to change through time. The method requires first of all that the model is expressed on a state space form. A state space model is characterized by a measurement equation and a transition equation. The Kalman filter is however only limited to linear models with Gaussian noises. Some non-linear filtering methods that are applicable to non-linear systems include the extended Kalman filter and the unscented Kalman filter. Particle filters can be applied to non-linear models with non-Gaussian noises. The main objective of this study is to use filtering to extract volatility from the information based asset pricing model as proposed in [4], referred to as the BHM model. Since the system of equations in the model are non-linear and gaussian, the extended kalman filter will be used. This study extends the work by [16] who applied filtering to the Heston Model and [5] who applied filtering to the Double Heston Model. A discrete dynamical system is considered with unobservable state vector x_k, for $k = 1, 2, \ldots.$, where k denotes time

$$x_k = f_k(x_{k-1}, w_k) \tag{1.1}$$

and f_k is a possibly non-linear and time-dependent function that represents the evolution of the state

process x_k. The state process is driven by noise denoted by w_k.

Suppose that an observable vector y_k at time k is also given such that:

$$y_k = h_k(x_k, v_k) \tag{1.2}$$

where h_k is a possibly non-linear and time-dependent function that defines the measurement y_k . The observations noise is denoted by v_k. The state process in equation 1.1 is called the state transition equation and the observation process in equation 1.2 is called measurement equation. The study begins by looking at filtering techniques starting with the kalman filter and extending it to the extended kalman filter. The Heston and BHM SVMs are then looked at and the state space representations derived. These state space models are used to obtain the measurement and transition equation used in the extended kalman filter to extract volatility from the BHM model.

2. Filtering Techniques

Kalman Filter

Kalman filter is only optimal for linear systems.

Given that the state function f_k from Equation 1.1 and the measurement function h_k from the Equation 1.2 are linear and their corresponding noise w_k and v_k respectively, are normally distributed and additive. Equation 1.1 can be expressed as

$$x_k = M_k x_{k-1} + w_k \tag{2.1}$$

and Equation 1.2 becomes

$$y_k = H_k x_k + v_k \tag{2.2}$$

The matrix M_k is assumed to be known, it defines the state transition evolution, and the matrix H_k defines the measurement process which is also assumed to be known. An assumption is made that the state noise $w_k \sim N(0, Q_k)$ and the measurement noise $v_k \sim N(0, R_k)$ are uncorrelated Gaussian random variables. In addition, w_k, v_k are independent of x_k, y_k respectively. By substituting x_k and y_k from Equations 2.1 and 2.2 in Equations 1.3 and 1.5, the computations result in the Kalman filtering algorithm with distributions given as:

$p(x_k|x_{k-1}) \sim N(M_k x_{k-1}, Q_k)$,

$p(y_k|x_k) \sim N(H_k x_k, R_k)$.

The objective is to find an estimate of the state vector x_k given the observations y_k. An estimate of the state vector x_k is obtained from the past estimated states, $\hat{x}_{k-1}$ in the prediction step. We have that

$\hat{x}_{k|\,k-1} = M_k \hat{x}_{k-1}$

$k|\,k-1$ represents the estimated state of the state x_k using previous $(k-1)$ estimated states.

$k|\ k$ represents the estimates of x_k using estimated states at $k|\ k-1$.

The computation of the past states is conducted by making use of the expectation of x_k given in Equation 2.1.

The error in estimation error is obtained from

$$e_k^- = x_k - \hat{x}_{k|\ k-1} \tag{2.4}$$

and the estimate error covariance

$$P_k^- = [e_k^- e_k^{-T}] \tag{2.5}$$

The prediction of the observations is computed from

$$\hat{y}_k = H_k \hat{x}_{k|\ k-1} \tag{2.7}$$

The estimate of $\hat{x}_{k|\ k}$ is obtained from $\hat{x}_{k|\ k-1}$ and a measurement residual weighted by Kalman gain K_k in the update step as

$$\hat{x}_{k|\ k} = \hat{x}_{k|\ k-1} + K_k(y_k - \hat{y}_k)$$

The measurement residual is computed from $b_k = y_k - \hat{y}_k$. The estimate error is

$$e_k = x_k - \hat{x}_{k|\ k}$$

and the estimate error covariance

$$P_k = [e_k e_k^T]$$

Extended Kalman Filter

This is an extension of the Kalman filter and is used where the dynamical systems are non-linear. Consider a case where the state transition function f_k and the observation function h_k given in Equations Equations 1.1 and 1.2 respectively are both non-linear and their corresponding noises are uncorrelated Gaussian random variables with $w_k \sim N(0, Q_k)$ and $v_k \sim N(0, R_k)$. Given that the densities in 1.3 and 1.5 are normally distributed, then the extended Kalman filter can be applied to obtain an estimate of the state v vector x_k given the observations y_k at time step k. In the extended Kalman filter algorithm, the states in the first step are predicted as:

$$\hat{x}_{k|\ k-1} = f_k(\hat{x}_{k-1}, 0) \tag{2.12}$$

The non-linear functions in the state transition and measurement equations are linearized to obtain the covariance using Jacobian matrices:

$$A_{ij} = \frac{\partial f_i(\hat{x}_{k-1}, 0)}{\partial x_j}, \qquad W_{ij} = \frac{\partial f_i(\hat{x}_{k-1}, 0)}{\partial w_j}$$

$$H_{ij} = \frac{\partial h_i\left(\hat{x}_{k|\,k-1}, 0\right)}{\partial x_j}, \qquad U_{ij} = \frac{\partial f_i\left(\hat{x}_{k|\,k-1}, 0\right)}{\partial v_j}$$

The predicted state covariance is thus

$$P_k^- = A_k P_{k-1} A_k^T + W_k Q_{k-1} W_k^T$$

Prediction of the measurement is given by

$$\hat{y}_k = h_k\left(\hat{x}_{k|\,k-1}, 0\right)$$

with covariance

$$F_k = H_k P_k^- H_k^T + U_k R_k U_k^T$$

The state vector x_k is estimated using the predicted states $\hat{x}_{k|\,k-1}$ in the update step. The measurement residual is weighted by the Kalman gain K_k,

$$\hat{x}_{k|\,k} = \hat{x}_{k|\,k-1} + K_k b_k$$

where the measurement residual is $b_k = y_k - \hat{y}_k$.

The optimal gain is

$$K_k = P_k^- H_k^T (F_k)^{-1}$$

and the updated covariance

$$P_k = P_k^- - K_k H_k P_k^-$$

3. Stochastic Volatility Models

After the October 1987 stock market crash, significant variations from normality have shown up in the term structure of volatility. Various academicians and traders have taken a keen interest on this observation and as a result, a lot of work has been done on this area. The danger of models used for pricing based on an incorrect assumption of log-normality is the risk of obtaining biased prices.

The Black-Scholes model which has been used extensively in the past is considered to be successful in asset pricing both in terms of approach and applicability. With the assumption of geometric brownian motion, the risk neutral density for the underlying assets is taken to be lognormal. Asset prices are often observed to have random volatility. These observations cannot be accurately be assumed to have a lognormal density since the density functions are fat-tailed and skewed. Stochastic Volatility Models (SVMs) can be used to address this weakness of the Black-Scholes Model. SVMs are models in which the variance of a stochastic process is itself randomly distributed. They are widely used in the finance industry for derivative pricing and hedging. The Heston model is an example of a SVM. It makes the assumption of stochastic volatility in the pricing of European call option and obtains a closed-form solution. The model further assumes that the volatility and the underlying asset price are correlated. In so doing, the Heston model is able to capture various properties of the financial information which the Black-Scholes model doesn't. However, the reliability of the model is questionable because the assumptions on the volatility and the underlying asset price dynamics display an ad-hoc nature. Another approach suggested later by Brody, Hughson and Macrina(BHM) obtains the asset price dynamics using a more realistic approach towards the structure of the market unlike the Heston model where the dynamics of the volatility and price are pre-specified. The approach is based on the assumption that the market information is incomplete. It specifies a model for the structure of the information available in the market since asset prices are determined by expectations on the future cash flows given the market information available. The BHM approach doesn't assume any dynamic model for the asset prices, it's observed that the asset price dynamics derived with the assumed information structure naturally has stochastic volatility giving a different view of the volatility nature. Accordingly, the model illustrates that the volatility of volatility is stochastic. The study shows that the Black Scholes asset pricing model can be obtained as a special case of the BHM model following the approach by [1]. A Black Scholes asset pricing model from an information-based perspective has been developed by [1], this is known as the BS-BHM model. A different model is obtained by [2] by applying Gaussian integral on the BS-BHM model, this is referred to as the BS-BHM-Updated model. It is based on the BlackScholes model from information-based perspective by Brody Hughston Macrina that it is updated in the results of Gaussian Integrals, more specifically on the analysis of algebra trick of completing square. Here, the Heston model and the BHM model are looked at in detail. Their state space representations are also presented which are then used in the filtering to estimate the volatilities.

The Heston Model

In this section, we first present the dynamic system for the Heston model under a risk-neutral measure $\mathbb{Q}$. The model in [7] assumes that an underlying stock price, S_t has a stochastic variance, V_t, that follows a CIR process. This process is represented by the following dynamical system:

$$dS_t = (r-q)S_t dt + \sqrt{V_t} S_t dW_t \qquad (3.1)$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dZ_t \qquad (3.2)$$

where r is a constant risk-free interest rate and q is a constant dividend. All the parameters κ, θ and σ are positive constant. The terms W_t and Z_t are Wiener processes that must be correlated with each other, that is;

$$(dW_t dZ_t) = \rho dt \qquad (3.3)$$

where ρ is the correlation coefficient between the return of the underlying asset and the changes in the variance.

The correlation, which is often negative, will ensure that the volatility for example will rise if the underlying asset value falls dramatically. In addition the variance is also mean-reverting, which is also evident in the market. The mean-reverting process is the term $\kappa(\theta - v)$. For option valuation, the approach by [9] is followed, such that the characteristic function of log returns $x_k = ln(S_k/S_{k-1})$ $(for\ k \leq t)$ of the Heston model is derived using the so called the little Heston trap. This characteristic function is only slightly different from the original formulation of [7], but it provides a better computation of the numerical integration. The European call option price under the Heston model in the one dimensional framework is given by;

$$C(S, V, K, \tau) = S_k e^{-q\tau} P_1 - K e^{-r\tau} P_2 \qquad (3.4)$$

Where $P_j(j = 1,2)$ are the risk-adjusted probabilities of the log of the underlying price $x_t = \ln(S_t/(S_{t-1})$. K denotes the strike price.

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\left[\frac{e^{-i\emptyset \ln K} f_j(\emptyset; x_k, V_k)}{i\emptyset}\right] d\emptyset \qquad (3.5)$$

for $j = 1, 2$.

The characteristic functions $f_j(\emptyset; x_k, V_k)$ in the probabilities are given by

$$f_j(\emptyset; x_k, V_k) = e^{i\emptyset \ln K + A_j(\emptyset,\tau) + B_j(\emptyset,\tau)V_k} \qquad (3.6)$$

where

$$B_j(\emptyset, \tau) = \frac{b_j - \rho\sigma\phi i + d_j}{\sigma^2}\left[\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right],$$

$$A_j(\emptyset, \tau) = r\phi i\tau + \frac{a}{\sigma^2}\left[\left(b_j - \rho\sigma\phi i + d_j\right)\tau - 2\ ln\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right)\right],$$

$$g_j = \frac{b_j - \rho\sigma\phi i + d_j}{b_j - \rho\sigma\phi i + d_j},$$

$$d_j = \sqrt{\left(\rho\sigma\phi i - b_j\right)^2 - \sigma^2\left(2u_j\phi i - \phi^2\right)}$$

And $i = \sqrt{-1}$, $\tau = T - k$, $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$, $a = \kappa\theta$, $b_1 = \kappa - \rho\sigma$, $b_2 = \kappa$ and ϕ is called the integration variable or node.

4. The BHM Model

The BHM Model as presented in [4] views asset price movements as an emergent phenomenon. The model's basis is pricing of assets by modelling the flow of market information. The market information in this case relates to the given assets expected future cash flows. It is different from the other models used in pricing assets mainly because the stochastic process governing the underlying asset price dynamics is deduced rather than being imposed in an arbitrary way. The asset price dynamics from the BHM model are used to determine the asset price dynamics in the BS-BHM Updated model. The BS-BHM-Updated model as presented by [2] uses Gaussian integrals to determine the equation of the asset pricing model. The result obtained is different from that of BS-BHM model due to an imprecision made in the Gaussian Integrals by BHM. Consider a single cash flow occurring at time T, represented by a random variable, X_T. The value of this variable will be revealed at time T. The flow of market information available to market participants is assumed to be contained in a process $\{\xi_t\}_{0\leq t\leq T}$ given by:

$$\xi_t = \sigma t X_T + \beta_{tT} \qquad (3.11)$$

ξ_t denotes the market information process, its also known as the information process. $\sigma t X_T$ contains the 'true information' about the value of the cash flow X_T. σ denotes the rate at which the true value of X_T is revealed to the market participants. $\{\beta_{tT}\}_{0\leq t\leq T}$ denotes a standard Brownian bridge over the interval $[0, T]$ with $\beta_{0T} = 0$ and $\beta_{TT} = 0$.

$$\beta_{tT} \sim N\left(0, \frac{t(T-t)}{T}\right)$$

X_T and β_{tT} are assumed to be independent in the information-based framework. From the market's point of view, it is the process W_t that drives the asset price dynamics.

According to [4], the dynamics of the price process are given as:

$$dS_t = r_t S_t dt + \Gamma_{tT} dW_t$$

In this study, an assumption will be made that r_t is a constant which implies that $r_t = r$. Thus;

$$dS_t = r S_t dt + \Gamma_{tT} dW_t \qquad (3.12)$$

Where Γ_{tT} denotes the absolute volatility process:

$$\Gamma_{tT} = P_{tT} \frac{\sigma T}{T-t} V_t \qquad (3.13)$$

P_{tT} denotes the discount factor and r_t denotes the short rate. By using Gaussian integrals, the equation of asset pricing model is presented as in [2] in the BS-BHMUpdated model as follows:

$$S_t = S_0 \exp\left(rt - \frac{1}{2}\frac{\sigma^2\tau}{\sigma^2\tau+1}\nu^2 T + \frac{\sigma\tau\nu\sqrt{T}}{t(\sigma^2\tau+1)}\xi_t\right) \qquad (3.14)$$

where $\tau = \frac{tT}{T-t}$.

ν is the asset price volatility parameter. σ and ν cannot be observed directly and are assumed to be constants. The equation above gives a different final result as compared to that of BHM's Black-Scholes Model from an Information-Based Perspective;

$$S_t = P_{tT} S_0 \exp\left(rT - \frac{1}{2}\nu^2 T + \frac{1}{2}\frac{\nu\sqrt{T}}{\sigma^2\tau+1} + \frac{\sigma\tau\nu\sqrt{T}}{t(\sigma^2\tau+1)}\xi_t\right) \tag{3.15}$$

The difference arises due to an imprecision done by BHM using Gaussian Integrals. The authors in [3] show that the BS-BHM Updated model in equation 3.15 follows the lognormal distribution given as:

$$log\left(\frac{S_t}{S_0}\right) \sim N\left[rt - \frac{1}{2}\frac{\sigma^2\tau}{\sigma^2\tau+1}\nu^2 T, \left(\frac{\sigma\tau\nu\sqrt{T}}{t(\sigma^2\tau+1)}\right)^2\left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)\right] \tag{3.16}$$

Let $A = rt - \frac{1}{2}\frac{\sigma^2\tau}{\sigma^2\tau+1}\nu^2 T$ and $B^2 = \left(\frac{\sigma\tau\nu\sqrt{T}}{t(\sigma^2\tau+1)}\right)^2\left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)$. This implies that

$$log\left(\frac{S_t}{S_0}\right) \sim N[A, B^2]$$

$$S_t \sim N\left[S_0 e^A, {S_0}^2 e^{2A}\left(e^{B^2} - 1\right)\right]$$

Thus:

$$S_t = S_0 e^{A+BZ} \tag{3.17}$$

where Z denotes a standard normal random variable. The European call option price under the BS-BHM-Updated model at time t with expiration date T and strike price K takes the form

$$C = E_{\mathbb{Q}}[max\{S_t - K, 0\}] \tag{3.18}$$

where C denotes the European call price. Following a similar approach to [1], an assumption is made that there exists an established pricing kernel and the absence of arbitrage. These two assumptions ensure the existence of a unique risk neutral probability measure $\mathbb{Q}$. The BS-BHM-Updated model European call price formula takes a similar form to the Black-Scholes Model given as:

$$C = S_0\ \Phi(d_1) - Ke^{-\left(A+\frac{B^2}{2}\right)}\Phi(d_2) \tag{3.19}$$

where $d_1 = \frac{\log\left(\frac{S_0}{K}\right)+A}{B} + B$, $d_2 = d_1 - B$ and $\Phi(x) = P[Z \leq x]$, Z being a standard normal random variable.

The author in [1] used to obtain the dynamics for the volatility in the BHM model. Let:

$$X_{tT} = \mathbb{E}[X_T/\xi_t] = \int_0^\infty x\pi_t(x)dx \tag{3.20}$$

where $\pi_t(x)$ denotes the conditional probability density for the random variable X_T;

$$\pi_t(x) = \frac{d}{dx}\mathbb{Q}[X_T \leq x/\xi_t] \tag{3.21}$$

$$d\pi_t(x) = \frac{\sigma T}{T-t}(x - X_{tT})\pi_t(x)dW_t \tag{3.22}$$

The SDE for D_{tT} is given as

$$dX_{tT} = \frac{\sigma T}{T-t}V_t\left[\frac{1}{T-t}(\xi_t - \sigma T X_{tT})dt + d\xi_t\right]. \tag{3.23}$$

This leads to:

$$dX_{tT} = \frac{\sigma T}{T-t}V_t dW_t$$

V_t denotes the conditional variance of X_T.

$$V_t = \mathbb{E}[(X_T - \mathbb{E}[X_T])^2] \tag{3.24}$$

$$V_t = \int_0^\infty x^2\pi_t(x)dx - X_{tT}^2 \tag{3.25}$$

Using Ito's lemma and letting $\kappa_t = \mathbb{E}[(X_T - \mathbb{E}[X_T])^3]$, it follows that:

$$dV_t = \left(\frac{\sigma T}{T-t}\kappa_t dZ_t\right) - \left(\frac{\sigma T}{T-t}\right)^2 V_t^2 dt$$

where deterministic nonnegative process $\{v_t\}_{0\leq t\leq T}$ is defined by

$$v_t = \sigma_t + \frac{1}{T-t}\int_0^t \sigma_s ds, \tag{3.26}$$

If σ is a constant, then:

$$v_t = \frac{\sigma T}{T-t} \tag{3.27}$$

The dynamics for V_t are therefore given as:

$$dV_t = -v_t^2 V_t^2 dt + v_t \kappa_t dZ_t \tag{3.28}$$

Using the approach in [1],the Black-Scholes asset-price model can be recovered from the BHM model.The authors in [2] presented the equation of the asset price model in the BS-BHM-Updated model as given in equation 3.14, considering the case where $\sigma^2 T = 1$, the asset pricing model reduces to:

$$S_t = S_0 \, exp\left(rt - \frac{1}{2}v^2 t + v\xi_t\right) \qquad (3.29)$$

This takes a similar form to the Black-Scholes asset pricing model obtained by making use of the assumption that the underlying share price follows a geometric Brownian motion in [6]. By applying Ito's lemma to obtain the SDE for $log\ S_t$ and integration both sides from 0 to t it is found that:

$$S_t = S_0 \, exp\left(rt - \frac{1}{2}\sigma^2 t + \sigma B_t\right) \qquad (3.30)$$

Thus, the Black Scholes model can be obtained as a special case of the BHM model.

State Space Representation

The authors in [5,16] use a filtering approach to obtain an estimate for the volatilities in the case of the Heston model and the Double Heston model respectively. In this study, this approach is extended to the BHM model. V_k is the state variable for the BHM model which is unobserved. The option prices are taken to be the model observations and the variance processes are taken to be the transition equations. Therefore, in order to estimate the unobservable factors and the model's parameters, the relationship between the option prices and the underlying state variables is used. This is the relationship between the evolution of the measurement equations and the state transition equations. A system of the measurement and transition equations is called the state space representation of the model which is referred to as the state space model. The measurement noise and the state noise are correlated in the heston model. Cholesky decomposition is used to decorrelate the sources of randomness so as to ensure that for the filters, the process noise and measurement noise are uncorrelated. In order to formulate the models in the state space representation, the state transition equations and the measurement equations need to be specified. The state space model for the Heston model is presented first, then it's extended to the BHM model. The authors in [16] shows that if the spot prices S_k and option prices$C(S_k,K)$ are taken as the observations and the variance V_k as the state, then the measurement equations are represented by

$$y_k = \ln S_k = \ln S_{k-1} + \left(r - q - \frac{1}{2}V_{k-1}\right)\Delta k + \sqrt{V_{k-1}}\sqrt{\Delta k}W_{k-1}. \qquad (3.33)$$

$$y_k^0 = g(S_k, V_k, \Theta) + \epsilon_t^0 \qquad (3.34)$$

where y_k^0 is the observable option prices, with identical independent distributed measurement errors $\epsilon_t^0 \to N(0, \sigma_0^2)$, independent of W_k and Z_k, and $g(.)$ is the theoretical option price computed from the Heston model. The state transition equations are given by the variance processes

$$\begin{pmatrix} V_k \\ V_{k-\Delta k} \end{pmatrix} = \begin{pmatrix} \kappa\theta\Delta k \\ 0 \end{pmatrix} + \begin{pmatrix} 1-\kappa\Delta k & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} V_{k-\Delta k} \\ V_{k-2\Delta k} \end{pmatrix} + \begin{pmatrix} \sigma\sqrt{\Delta k V_{k-\Delta k}} \\ 0 \end{pmatrix} Z_{k-1}$$

Extending this approach to the BHM Model where the system of stochastic equations are given by equation 3.12 and equation 3.34. An assumption is made that the brownian motions, W_t and Z_t are uncorrelated. The asset price volatility is given as $\Gamma_{tT} = v_t P_{tT} V_t$ By discretizing equation 3.12, the following measurement equation is obtained:

$$S_k = S_{k-1} + rS_{k-1}\Delta k + v_{k-1}P_{(k-1)T}V_{k-1}\sqrt{\Delta k}W_{k-1} \tag{3.35}$$

The variance process is given by:

$$V_k = V_{k-1} - v_{k-1}^2 V_{k-1}^2 \Delta k + v_{k-1}\, \kappa_{k-1}\sqrt{\Delta k} Z_{k-1} \tag{3.36}$$

Thus the state transition equations are given by the variance processes:

$$\begin{pmatrix} V_k \\ V_{k-1} \end{pmatrix} = \begin{pmatrix} 1 - v_{k-1}^2 V_{k-1}\Delta k & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} V_{k-1} \\ V_{k-2} \end{pmatrix} + \begin{pmatrix} v_{k-1}\kappa_{k-1}\Delta k \\ 0 \end{pmatrix} Z_{k-1}$$

5. Filtering in Stochastic Volatility Models

In the previous sections, the extended Kalman filter as well as the stochastic volatility models have been discussed. Here, a similar approach to [16] where the non-linear filtering methods are applied to the Heston Model is used. The author in [5] also used a similar approach in applying non-linear filtering to the Double Heston Model. In this study, this approach is extended to the BHM Model. In particular, the extended kalman filter non-linear filtering method is used to extract volatility in the BHM model.

Filtering in the Heston Model

In the case of the Heston model, the initial mean is given by $\hat{x}_0 = V_0$ and the initial covariance $\hat{P} = diag(\sigma^2 \Delta k V_0, 0)$. V_0 denotes the initial variance level. The parameters to be estimated are $\kappa,\theta,\sigma,V_0,\rho$. These parameters are first estimated and the kalman filter is then used to estimate volatility. The Jacobian matrices in the extended Kalman filter are obtained as follows:

$$A_k = 1 - \kappa\Delta k. \tag{4.1}$$

$$W_k = \sigma\sqrt{\Delta k V_{k-1}} \tag{4.2}$$

Thus, the Jacobian matrices under the Heston model are given by equation 4.1 and equation 4.2. The matrix H_k can be viewed as the derivative of the call price with respect to the implied volatility. In the Heston model, the shape of the surface of the implied volatility is determined by the parameters which drive the variance process. In order to compute H_k, the derivative is computed based on two parameters, $v_1 = \sqrt{V_0}$ and $v_2 = \sqrt{\theta}$ which gives the following Jacobian matrices:

$$H_1 = \frac{\partial h\left(\hat{x}_{k|\,k-1}, 0\right)}{\partial v_1} = \frac{\partial h\left(\hat{x}_{k|\,k-1}, 0\right)}{\partial V_0} 2\sqrt{V_0} \tag{4.3}$$

$$H_2 = \frac{\partial h\left(\hat{x}_{k|\,k-1}, 0\right)}{\partial v_2} = \frac{\partial h\left(\hat{x}_{k|\,k-1}, 0\right)}{\partial \theta} 2\sqrt{\theta} \tag{4.4}$$

Substituting $h\left(\hat{x}_{k|\,k-1}, 0\right)$ for the value of a European call option under the Heston model, that is:

$$h\left(\hat{x}_{k|\,k-1}, 0\right) = Se^{-q\tau}P_1 - Ke^{-r\tau}P_2 \tag{4.5}$$

This results in:

$$H_1 = \frac{\partial(Se^{-q\tau}P_1 - Ke^{-r\tau}P_2)}{\partial V_0} 2\sqrt{V_0}$$

$$H_1 = Se^{-q\tau}\frac{\partial P_1}{\partial V_0} 2\sqrt{V_0} - Ke^{-r\tau}\frac{\partial P_2}{\partial V_0} 2\sqrt{V_0}$$

where

$$\frac{\partial P_j}{\partial V_0} = \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\emptyset \ln K} f_j(\emptyset; S_k, V_k)B_j(\tau, \emptyset)}{i\emptyset}\right] d\emptyset$$

For $j = 1{,}2$

$$H_2 = \frac{\partial(Se^{-q\tau}P_1 - Ke^{-r\tau}P_2)}{\partial \theta} 2\sqrt{\theta}$$

$$H_2 = Se^{-q\tau}\frac{\partial P_1}{\partial \theta} 2\sqrt{\theta} - Ke^{-r\tau}\frac{\partial P_2}{\partial \theta} 2\sqrt{\theta}$$

where

$$\frac{\partial P_j}{\partial \theta} = \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\emptyset \ln K} f_j(\emptyset; S_k, V_k)\partial A_j(\tau, \emptyset)/\partial\theta}{i\emptyset}\right] d\emptyset$$

and

$$\frac{\partial A_j(\tau, \emptyset)}{\partial \theta} = \frac{\kappa}{\sigma^2}\left[\left(b_j - \rho\sigma i\phi + d_j\right)\tau - 2ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right]$$

Filtering in the BHM Model

In this case, the mean is initialized as $\hat{x}_0 = V_0$ and the initial covariance is $P_0^- = diag(v_0^2\kappa_0\Delta k, 0)$. For the BHM model, the parameters to be estimated are σ,v and V_0.. The parameters are first initialized and the kalman filter is used to estimate volatility.

Using a similar approach to the Heston model, the Jacobian matrices in the extended Kalman filter for the BHM model can be obtained as follows:

$$A_k = 1 - 2v_{k-1}^2 V_{k-1}\Delta k \qquad (4.13)$$

$$W_k = v_{k-1}\kappa_{k-1}\sqrt{\Delta k} \qquad (4.14)$$

Thus, the Jacobian matrices under the BHM model are given by equation 4.13 and 4.14.

Under the BHM model, the matrix H_k is computed based on a parameter V_0 which gives:

$$H_1 = \frac{\partial h\left(\hat{x}_{k|\,k-1}, 0\right)}{\partial x_1} = \frac{\partial h\left(\hat{x}_{k|\,k-1}, 0\right)}{\partial V_0} \qquad (4.15$$

Substituting $h\left(\hat{x}_{k|\,k-1}, 0\right)$ for the value of a European call option under the BS-BHM-Updated model, that is:

$$h\left(\hat{x}_{k|\,k-1}, 0\right) = S_0\Phi(d_1) - Ke^{-\left(A+\frac{B^2}{2}\right)}\Phi(d_2)$$

This results in:

$$H_1 = \frac{\partial\left(S_0\Phi(d_1) - Ke^{-\left(A+\frac{B^2}{2}\right)}\Phi(d_2)\right)}{\partial V_0} 2\sqrt{V_0}$$

$$H_1 = S_0\frac{\partial\Phi(d_1)}{\partial V_0}2\sqrt{V_0} - K\frac{\partial e^{-\left(A+\frac{B^2}{2}\right)}\Phi(d_2)}{\partial V_0}2\sqrt{V_0}$$

$$H_2 = \frac{\partial\left(S_0\Phi(d_1) - Ke^{-\left(A+\frac{B^2}{2}\right)}\Phi(d_2)\right)}{\partial \theta} 2\sqrt{\theta}$$

6. Conclusion

In this study, the extended kalman filtering technique has been applied to extract volatility in the information based asset pricing framework. Using the approach by the authors in [1], the dynamics for the volatility process are obtained. The asset price dynamics in [3] are then used with the volatility process dynamics to obtain the state space model. The price obtained from the BS-BHM-Updated Model is used as the measurement equation and the variance process is used as the state transition equation.The state space model is then used to perform filtering using the extended Kalman filtering technique since the system of equations in the model are non-linear. The Brownian motion process is the same for the state transition equation and measurement equation under the BHM model. The European Call Option price obtained using the BS-BHM-Updated Model will be used to determine the option price. This option price is used to obtain the measurement equation.

7. Recommendation

Further studies can be done by using other non-linear filtering methods such as the unscented kalman filter to extract volatility in the BHM Model. Methods such as maximum likelihood can also be used to estimate the parameters used in the model. In addition, the case where the assumption that the interest rate is a constant can be relaxed.

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