International Journal of Sciences: Basic and Applied Research (IJSBAR)

ISSN 2307-4531
(Print & Online)

http://gssrr.org/index.php?journal=JournalOfBasicAndApplied

Deriving Formulas for Three Integers Such That the Sum of Their Squares and the Sum of Products of Their Squares are Both Perfect Square Integers

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Abstract

In this research study we will derive formulas for three integers whose squares sum and the sum of their squares products are both perfect square integers. That is if $x, y,$ and z are three integers, then $x^2 + y^2 + z^2$ and $x^2y^2 + x^2z^2 + y^2z^2$ are both perfect square integers. Proofs are most often arrived at through deductive reasoning. A proof is an argument where a given statement is true. In this proof the so called direct proof had used.

Keywords: Integers; Square integers; Perfect square integers; variables; algebraic operations.

1. Introduction

1.1 Background of the study

Proofs in mathematics are most often arrived at through deductive reasoning. A proof is an argument where a given statement is true.

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In the process of reasoning in deductive logic any proof involves some hypothesis and conclusion. A deductive logical system or an axiomatic system is any logical system having primitive terms and postulates. It consists of four essential parts. These are undefined terms; defined terms; axioms and theorems:

1.1.1 Undefined Terms

An axiomatic system is a formal system based on ultimately on primitive terms which have no definite meaning. For example, in algebra, “set” and “element” often are taken as undefined terms (concepts). All basic terms cannot be defined. We use some undefined terms to define other terms [1].

1.1.2 Defined Terms

All non-primitive technical terms used in the system are defined terms, describing by the use of undefined terms. A model of the axiomatic system is obtained by assigning terms of definite meaning in such a way as to make the postulates used true statements. A good definition should be:

- Consistent that is convey exactly the same meaning in every possible case;
- Clear and precise without redundant words;
- Broad enough to include all objects in the given set and yet detailed enough to exclude any that do not belong to the set [1].

1.1.3 Axioms

An axiom (assumption or postulate) is any statement that is assumed to be true for the system under consideration. It is unproven statement containing only primitive forms. It is an assumption of some basic property about the elements of the system. Thus, axioms are neither self-evident nor necessarily true. In deciding axioms, care must be taken to ensure that they are consistent that is they do not contradict themselves or lead to contradictions [1].

1.1.4 Theorems

A theorem is a statement which must be logically deduced or proved by using undefined terms, defined terms, axioms and statements whose truth value has been previously established. A theorem can be written as a conditional statement in the form “If …... then…..”

There are two parts in a theorem. These are the “if-part” and the “then-part”. The sentence between the “if and the then” or the if-part is called antecedent or hypothesis and the sentence that follows “then” (or the then-part) is called consequent or conclusion of the theorem. For example, in the conditional statement:

“If you study hard, then you will score good result”, the sentence between the “if” and “then” (you study hard) is called the antecedent or hypothesis and the sentence that follows the “then” (you will score good result) is called the consequent or conclusion. There are a variety of methods and techniques of proving statements depending on their logical terms. We only consider the following commonly used methods:

Direct Proof

In this method, one starts with the hypothesis of the theorem and then constructs a chain of reasoning where each step is justified by the laws of logic using axioms, definitions, or previously proved theorems to show that the conclusion is true. In this method, the statement is written in the form: $p \Rightarrow q$, so, to prove $p \Rightarrow q$ is true, it is sufficient to consider the case when p is true. Assume p is true, if we can arrive at a conclusion that q is also true, then the statement is proved [1].

Mathematical Induction

Many results were found in similar manner, that is, by extrapolating a general result from a few particular cases. That general result is correct can be often be proved by a method called induction. Mathematical induction is a form of deductive reasoning in which conclusions are established from assumptions. Mathematical induction proves a general statement involving positive integers. Let P_n denote a mathematical statement associated with each positive integer n, the following principles are require: Suppose there is a given statement involving the natural number n and that:

1. The statement is true for $n = 1$.

2. If the statement is true for $n = k$, where k is a natural number, then the statement is true $n = k + 1$. Thus, the statement is true for every natural number n [1]

Thus, by using direct proof we will prove that if $x, y,$ and z are three integers, then $x^2 + y^2 + z^2$ and $x^2y^2 + x^2z^2 + y^2z^2$ are both perfect square integers.

1.2 Statement of the problem

A variable is a symbol used to represent an unknown element of a set. A variable is a place holder or blank for the name of some members of the set. In most of our work in algebra the set whose member in variable represents is usually a set of numbers. In this research we will see how variables are used to represent important mathematical members or elements. In mathematics a perfect square is defined as: a number that can be expressed as the product of two equal integers. Examples: $4,\ 9,\ 16, 25, 36, \cdots$ are perfect square integers [2] The research problem can be restated as if $x, y,$ and z are three integers, then $x^2 + y^2 + z^2$ and $x^2y^2 + x^2z^2 + y^2z^2$ are both perfect square integers. We will derive formulas for each of the three integers with the required conditions.

1.3 Objective of the study

- Deriving formulas for three integers such that the sum of their squares and the sum of products of their squares are both perfect square integers.

2. Materials and Methods of the study

2.1 Research design of the study

- In this research study survey design was used.

2.2 Methods of data analysis

- Descriptive Data Analysis was used.

3. Results and Discussions of the study

Let us change the word statement to algebraic expressions using ordinary basic algebra operations like addition $(+)$, multiplication $(\times)$, subtraction $(-)$ and division $(\div)$.

Let $x^2 + y^2 + z^2 = S^2$ and $x^2y^2 + x^2z^2 + y^2z^2 = T^2$, where S and T are integers.

Let $x = p^2 + q^2 - r^2, y = 2pr, z = 2qr$, where p, q and r are integers.

To express the integers p, q, r in terms of single integer n and finally express the integers x, y, z in terms of n. Then

$S^2 = x^2 + y^2 + z^2 = (p^2 + q^2 - r^2)^2 + (2pr)^2 + (2qr)^2$ ………… Substituting for x, y, z

$= p^4 + 2p^2q^2 - 2p^2r^2 - 2q^2r^2 + q^4 + r^4 + 4p^2r^2 + 4q^2r^2$ …… Squaring out the squares

$= p^4 + 2p^2q^2 + 2p^2r^2 + 2q^2r^2 + q^4 + r^4$ …………….. Collecting the like terms together

$= (p^2 + q^2 + r^2)^2$ ………………………….. Writing in contracted form

$$\therefore S^2 = (p^2 + q^2 + r^2)^2$$

$T^2 = x^2y^2 + x^2z^2 + y^2z^2$

$= (p^2 + q^2 - r^2)^2(2pr)^2 + (p^2 + q^2 - r^2)^2(2qr)^2 + (2pr)^2(2qr)^2$ ……………………….. Substituting

$= 4p^2r^2(p^2 + q^2 - r^2)^2 + 4q^2r^2(p^2 + q^2 - r^2)^2 + 16p^2q^2r^4$ …………… (Multiplication is commutative)

$= (4p^2r^2 + 4q^2r^2)(p^2 + q^2 - r^2)^2 + 16p^2q^2r^4$ ………………………………………………. (Factoring)

$= 4r^2[p^2 + q^2](p^2 + q^2 - r^2)^2 + 16p^2q^2r^4$ ………………………….. (Taking the GCF of the terms)

$$\therefore T^2 = 4r^2[p^2 + q^2](p^2 + q^2 - r^2)^2 + 16p^2q^2r^4$$

Dividing both sides of the equation by $4r^2 \neq 0$:

$\frac{T^2}{4r^2} = [p^2 + q^2](p^2 + q^2 - r^2)^2 + 4p^2q^2r^2$.

$\left(\frac{T}{2r}\right)^{\mathbf{2}} = [p^2 + q^2](p^2 + q^2 - r^2)^2 + 4p^2q^2r^2$…………………………………… Writing in contracted form

This equation is equivalent with the following equation:

$\left(\frac{T}{2r}\right)^{\mathbf{2}} = p^6 + p^2q^4 + p^2r^4 + 2p^4q^2 - 2p^4r^2 + 2p^2q^2r^2 + q^2p^4 + q^6 + q^2r^4 + 2p^2q^4 - 2q^2p^2r^2 - 2q^4r^2$
………………………. ……………………………………. Expanding the right hand side of the equation

Critically looking this equation in terms of the variables p, q, and r, (integers), we can clearly see that the right hand side is an equation of degree 6 in both p and q. But, degree 4 in r. To change this equation to degree 4 both in p and q, we need to remove r from the equation. So, let $r = p - nq$, where n is an integer. Then

$\left(\frac{T}{2(p-nq)}\right)^{\mathbf{2}} = [p^2 + q^2](p^2 + q^2 - (p - nq)^2)^2 + 4p^2q^2(p - nq)^2$………………………………………… *

Now, we will look for equivalent expression of $p^2 + q^2 - (p - nq)^2$ from equation *.

Since $r = p - nq$. Then

$p^2 + q^2 - (p - nq)^2 = p^2 + q^2 - (p^2 - 2pnq + n^2q^2)$ ………………………………. Squaring out the square

$= p^2 + q^2 - p^2 + 2pnq - n^2q^2$ …………………………………… Removing the bracket

$= q^2 + 2pnq - n^2q^2$ (Since $p^2 - p^2 = 0$) ………………………… Simplifying like terms

$= q[2pn + (1 - n^2)q]$ …………………….. ………………………………. (Taking-Out the GCF of the terms)

$\therefore p^2 + q^2 - (p - nq)^2 = q[2pn + (1 - n^2)q]$

Now, let us substitute to equation * above:

$\left(\frac{T}{2(p-nq)}\right)^{\mathbf{2}} = [p^2 + q^2][q(2pn + (1 - n^2)q)]^2 + 4p^2q^2(p - nq)^2$. Then let's divide by $q^2 \neq 0$, both sides of the equation:

$\left(\frac{T}{2q(p-nq)}\right)^{\mathbf{2}} = [p^2 + q^2][(2pn + (1 - n^2)q)]^2 + 4p^2(p - nq)^2$.

Let $R = \frac{T}{2q(p-nq)}$. Then

$$\therefore R^{2} = [p^2 + q^2][(2pn + (1 - n^2)q)]^2 + 4p^2(p - nq)^2$$

Let us summarize what we have obtained so far:

- $T = 2q(p - nq)R$
- $R^{2} = [p^2 + q^2][(2pn + (1 - n^2)q)]^2 + 4p^2(p - nq)^2$
- $x = p^2 + q^2 - r^2, y = 2pr, z = 2qr$

Let us now expand R^{2}:

$$R^{2} = [p^2 + q^2][(2pn + (1 - n^2)q)]^2 + 4p^2(p - nq)^2$$

$$= p^2[(2pn + (1 - n^2)q)]^2 + q^2[(2pn + (1 - n^2)q)]^2 + 4p^2(p - nq)^2$$

$$= p^2[4p^2n^2 + 4pnq(1 - n^2) + q^2(1 - n^2)^2] + q^2[4p^2n^2 + 4pnq(1 - n^2) + q^2(1 - n^2)^2] + 4p^2(p^2 - 2pnq + n^2q^2)$$

$$= p^2[4p^2n^2 + 4pnq - 4pn^3q + q^2(1 - 2n^2 + n^4)] + q^2[4p^2n^2 + 4pnq - 4pn^3q + q^2(1 - 2n^2 + n^4)] + 4p^4 - 8p^3nq + 4p^2n^2q^2$$

$$= p^2[4p^2n^2 + 4pnq - 4pn^3q + q^2 - 2n^2q^2 + n^4q^2] + q^2[4p^2n^2 + 4pnq - 4pn^3q + q^2 - 2n^2q^2 + n^4q^2] + 4p^4 - 8p^3nq + 4p^2n^2q^2$$

$$= [4p^4n^2 + 4p^3nq - 4p^3n^3q + q^2p^2 - 2n^2q^2p^2 + n^4q^2p^2] + [4p^2n^2q^2 + 4pnq^3 - 4pn^3q^3 + q^4 - 2n^2q^4 + n^4q^4] + 4p^4 - 8p^3nq + 4p^2n^2q^2$$

$$= (4p^4 + 4p^4n^2) + (4p^3nq - 4p^3n^3q - 8p^3nq) + (q^2p^2 - 2n^2q^2p^2 + n^4q^2p^2 + 4p^2n^2q^2 + 4p^2n^2q^2) + (4pnq^3 - 4pn^3q^3) + (q^4 - 2n^2q^4 + n^4q^4)$$

$$= 4(1 + n^2)p^4 - 4n(1 + n^2)p^3q + (1 + 6n^2 + n^4)p^2q^2 + 4n(1 - n^2)pq^3 + (1 - n^2)^2q^4$$

$$\therefore R^{2} = 4(1 + n^2)p^4 - 4n(1 + n^2)p^3q + (1 + 6n^2 + n^4)p^2q^2 + 4n(1 - n^2)pq^3 + (1 - n^2)^2q^4$$

Now, R^{2} is of degree 4 in both p and q.

Let $R = \alpha p^2 + 2npq + (1 - n^2)q^2$ is of degree 2 in both p and q. Then

$R^2 = [\alpha p^2 + 2npq + (1 - n^2)q^2]^2$

$$= \alpha^2p^4 + 4\alpha p^3nq + 2\alpha p^2q^2(1 - n^2) + 4npq^3(1 - n^2) + (1 - n^2)^2q^4 + 4n^2p^2q^2$$

$$= \alpha^2(p^4) + 4\alpha n(p^3q) + (4n^2 + 2\alpha(1-n^2))p^2q^2 + 4n(1-n^2)pq^3 + (1-n^2)^2q^4$$

$$\therefore R^2 = \alpha^2(p^4) + 4\alpha n(p^3q) + (4n^2 + 2\alpha(1-n^2))p^2q^2 + 4n(1-n^2)pq^3 + (1-n^2)^2q^4$$

Now let us consider the following two equations simultaneously:

- $R^2 = 4(1+n^2)p^4 - 4n(1+n^2)p^3q + (1+6n^2+n^4)p^2q^2 + 4n(1-n^2)pq^3 + (1-n^2)^2q^4$
- $R^2 = \alpha^2(p^4) + 4\alpha n(p^3q) + (4n^2 + 2\alpha(1-n^2))p^2q^2 + 4n(1-n^2)pq^3 + (1-n^2)^2q^4$

Note that the above two equations are said to be equal if and only if the coefficients of similar terms are equal. So that considering corresponding coefficients of p^2q^2 to be equal, we let

$$1 + 6n^2 + n^4 = 4n^2 + 2\alpha(1-n^2)$$

$$1 + 6n^2 + n^4 - 4n^2 = 2\alpha(1-n^2)$$

$$2\alpha(1-n^2) = 1 + 6n^2 + n^4 - 4n^2$$

$$2\alpha(1-n^2) = 1 + 6n^2 - 4n^2 + n^4$$

$$2\alpha(1-n^2) = 1 + 2n^2 + n^4$$

$$\alpha = \frac{1 + 2n^2 + n^4}{2(1-n^2)}$$

$$\alpha = \frac{(1+n^2)^2}{2(1-n^2)},\ n \neq 1$$

Then remaining terms are equal so that

$$4(1+n^2)p^4 - 4n(1+n^2)p^3q = \left(\frac{(1+n^2)^2}{2(1-n^2)}\right)^2 p^4 + 4\frac{(1+n^2)^2}{2(1-n^2)}np^3q$$

$4p^4 - 4np^3q = \frac{(1+n^2)^3}{4(1-n^2)^2}p^4 + 2\frac{(1+n^2)}{(1-n^2)}np^3q$ Dividing both sides by $(1+n^2)$ and collecting the like terms together we have:

$$4p^4 - \frac{(1+n^2)^3}{4(1-n^2)^2}p^4 = 2\frac{(1+n^2)}{(1-n^2)}np^3q + 4np^3q$$

$$\left(4 - \frac{(1+n^2)^3}{4(1-n^2)^2}\right)p^4 = \left(2\frac{(1+n^2)}{(1-n^2)}n + 4n\right)p^3q$$

$$\left(\frac{16(1-n^2)^2-(1+n^2)^3}{4(1-n^2)^2}\right)p^4=\left(\frac{2n(1+n^2)+4n(1-n^2)}{(1-n^2)}\right)p^3q$$

$$\left(\frac{16[1-2n^2+n^4]-(1+2n^2+n^4+n^2+2n^4+n^6)}{4(1-n^2)}\right)p^4=\left(\frac{2n(1+n^2)+4n(1-n^2)}{1}\right)p^3q$$

$$\left(\frac{16-32n^2+16n^4-1-2n^2-n^4-n^2-2n^4-n^6}{4(1-n^2)}\right)p=\left(\frac{2n(1+n^2)+4n(1-n^2)}{1}\right)q$$

$$[15-35n^2+13n^4-n^6]p=4(1-n^2)[6n-2n^3]q$$

Solving for $\frac{p}{q}$:

$$\frac{p}{q}=\frac{4(1-n^2)[6n-2n^3]}{15-35n^2+13n^4-n^6}$$

$$\frac{p}{q}=\frac{8n(1-n^2)[3-n^2]}{15-35n^2+13n^4-n^6}$$

Since $15-35n^2+13n^4-n^6=3n^4-30n^2+15-n^6+10n^4-5n^2=3(n^4-10n^2+5)-n^2(n^4-10n^2+5)=(3-n^2)(n^4-10n^2+5)$.

$$\frac{p}{q}=\frac{8n(1-n^2)[3-n^2]}{15-35n^2+13n^4-n^6}=\frac{8n(1-n^2)[3-n^2]}{(3-n^2)(n^4-10n^2+5)}=\frac{8n(1-n^2)}{(n^4-10n^2+5)}.$$

Thus, $p=8n(1-n^2)$, $q=n^4-10n^2+5$.

Note that $r=p-nq$.

$$=8n(1-n^2)-n(n^4-10n^2+5)$$

$$=8n-8n^3-n^5+10n^3-5n$$

$$=3n+2n^3-n^5$$

$$r=n(3+2n^2-n^4)$$

$$\therefore p=8n(1-n^2),\ q=n^4-10n^2+5,\ r=n(3+2n^2-n^4)$$

Substituting this final result in the equations $x=p^2+q^2-r^2, y=2pr, z=2qr$ for solving in terms of n, we have obtained the following formulas and the derivation of the formulas has completed:

$$x=p^2+q^2-r^2=(8n(1-n^2))^2+(n^4-10n^2+5)^2-(n(3+2n^2-n^4))^2$$
$$y=2pr=2(8n(1-n^2))(n(3+2n^2-n^4))$$

$$z = 2qr = 2(n^4 - 10n^2 + 5)(n(3 + 2n^2 - n^4))$$

Practical Examples:

Let us see the following examples using the principle of mathematical induction and the derived formulas:

Case 1: Let $n = 2$. Then find the three integers.

Step 1: - Find p, q, r

$$p = 8n(1 - n^2) = 8 \times 2(1 - 2^2) = -48.$$

$$q = n^4 - 10n^2 + 5 = (2)^4 - 10(2)^2 + 5 = 16 - 40 + 5 = -19.$$

$$r = n(3 + 2n^2 - n^4)$$

$$= 2(3 + 2(2)^2 - (2)^4)$$

$$= 2(3 + 2 \times 4 - 16).$$

$$= 2(3 + 8 - 16)$$

$$= 2(11 - 16)$$

$$= 2(-5)$$

$$= -10$$

$$\therefore p = -48, q = -19, r = -10.$$

Step 2:- Find x, y, z

- $x = p^2 + q^2 - r^2$

$$= (-48)^2 + (-19)^2 - (-10)^2$$

$$= 2304 + 361 - 100$$

$$= 2565$$

- $y = 2pr$

$$= 2 \times (-48) \times (-10)$$

$= 960$

- $z = 2qr$

$= 2 \times (-19) \times (-10)$

$= 380$

For simplicity divide each x, y, z by 5 (Since each is a multiple of 5)

$x = 513, y = 192, z = 76.$

Step 3: - Check that

i. $x^2 + y^2 + z^2$ is a perfect square integer.
ii. $x^2y^2 + y^2z^2 + x^2z^2$ is a perfect square integer.

Then

$x^2 + y^2 + z^2 = (513)^2 + (192)^2 + (76)^2$

$$= 263169 + 36864 + 5776$$

$$= 305{,}809$$

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{305{,}809}$$

$= 553$ is a square root

Hence, $x^2 + y^2 + z^2$ is a perfect square integer.

$$x^2y^2 + x^2z^2 + y^2z^2 = (513)^2(192)^2 + (513)^2(76)^2 + (192)^2(76)^2$$

$$= (263169)(36864) + (263169)(5776) + (36864)5776$$

$$= 9701462016 + 1520064144 + 212926464$$

$$= 11434452624$$

$$\sqrt{x^2y^2 + x^2z^2 + y^2z^2} = \sqrt{11434452624}$$

$= 106932$ is a square root

Hence, $x^2y^2 + x^2z^2 + y^2z^2$ is a perfect square integer.

$\therefore x^2 + y^2 + z^2$ and $x^2y^2 + x^2z^2 + y^2z^2$ are both perfect square integers.

Thus, we have proved that the sum of the squares and the sum of the products of their squares are both perfect square integers for $n = 2$.

$\therefore x = 513, y = 192, z = 76$ are the three required integers.

Case 2: Let $n = 3$. Determine the three integers. Then

Step 1: - Find p, q, r

$$p = 8n(1 - 3^2) = 8 \times 3(1 - 3^2) = 24(1 - 9).$$

$$= 24(-8)$$

$$= -192$$

$$q = n^4 - 10n^2 + 5 = (3)^4 - 10(3)^2 + 5$$

$$= 81 - 90 + 5$$

$$= 86 - 90.$$

$$= -4$$

$$r = n(3 + 2n^2 - n^4)$$

$$= 3(3 + 2(3)^2 - (3)^4)$$

$$= 3(3 + 2 \times 9 - 81).$$

$$= 3(3 + 18 - 81)$$

$$= 3(21 - 81)$$

$$= 3(-60)$$

$$= -180$$

$$\therefore p = -192, q = -4, r = -180.$$

Step 2:- Find x, y, z

- $x = p^2 + q^2 - r^2$

$= (-192)^2 + (-4)^2 - (-180)^2$

$= 36864 + 16 - 32400$

$= 36880 - 32400$

$= 4480$

- $y = 2pr$

$= 2 \times (-192) \times (-180)$

$= 69120$

- $z = 2qr$

$= 2 \times (-4) \times (-180)$

$= 1440$

$\therefore x = 4480, y = 69120, z = 1440.$

Step 3: - Check that

iii. $x^2 + y^2 + z^2$ is a perfect square integer.

iv. $x^2y^2 + x^2z^2 + y^2z^2$ is a perfect square integer. Then

$x^2 + y^2 + z^2 = (4480)^2 + (69120)^2 + (1440)^2$

$= 20070400 + 4777574400 + 2073600$

$= 4799718400$

$\sqrt{x^2 + y^2 + z^2} = \sqrt{4799718400}$

$= 69280$ is a square root

Hence, $x^2 + y^2 + z^2$ is a perfect square integer.

$x^2y^2 + y^2z^2 + x^2z^2 = (4480)^2(69120)^2 + (4480)^2(1440)^2 + (69120)^2(1440)^2$

$= (20070400)(477574400) + (20070400)(2073600) + (4777574400)(2073600)$

$= 105836225495040000$

$$\sqrt{x^2y^2 + y^2z^2 + x^2z^2} = \sqrt{105836225495040000}$$

$= 325324800$ is a square root.

Hence, $x^2y^2 + y^2z^2 + x^2z^2$ is a perfect square integer.

$\therefore x^2 + y^2 + z^2$ and $x^2y^2 + x^2z^2 + y^2z^2$ are both perfect square integers.

Thus, we have proved that the sum of the squares and the sum of the squares products are both perfect square integers for $n = 3$.

$\therefore x = 4480, y = 69120, z = 1440$ are the three required integers

<u>Case 3</u>: Let $n = 4$. Find the three integers. Then

Step 1: Find p, q, r

$p = 8 \times 4(1 - 4^2) = 32(1 - 4^2) = 32(1 - 16).$

$= 32(-15)$

$= -480$

$q = n^4 - 10n^2 + 5 = (4)^4 - 10(4)^2 + 5$

$= 256 - 160 + 5$

$= 261 - 90.$

$= 171$

$r = n(3 + 2n^2 - n^4)$

$= 4(3 + 2(4)^2 - (4)^4)$

$= 4(3 + 2 \times 16 - 256).$

$= 4(3 + 32 - 256)$

$= 4(35 - 256)$

$= 4(-221)$

$= -884$

$\therefore p = -480, q = 171, r = -884.$

Step 2: Find x, y, z

- $x = p^2 + q^2 - r^2$

$= (-480)^2 + 171)^2 - (-884)^2$

$= 230400 + 29241 - 781456$

$= 259641 - 781456$

$= -521815$

- $y = 2pr$

$= 2 \times (-480) \times (-884)$

$= 848640$

- $z = 2qr$

$= 2 \times (171) \times (-884)$

$= -302328$

$\therefore x = -521815, y = 848640, z = -302328.$

Step 3: Check that

v. $x^2 + y^2 + z^2$ is a perfect square integer.

vi. $x^2y^2 + x^2z^2 + y^2z^2$ is a perfect square integer.

Then

$x^2 + y^2 + z^2 = (-521815)^2 + (848640)^2 + (-302328)^2$

$$= 272290894225 + 720189849600 + 91402219584$$

$$= 1083882963409$$

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{1083882963409}$$

$= 1041097$ is a square root

Hence, $x^2 + y^2 + z^2$ is a perfect square integer.

$$x^2y^2 + y^2z^2 + x^2z^2 = (4480)^2(69120)^2 + (4480)^2(1440)^2 + (69120)^2(1440)^2$$

$$= (272290894225)(720189849600) + (720189849600)(91402219584) + (272290894225)(91402219584)$$

$$= 196101138159352300000000 + 65826950775307130000000 + 24887992104677170000000$$

$$= 196101138159352300000000 + 65826950775307130000000 + 24887992104677170000000 = 110325056695919500000000$$

$$\sqrt{x^2y^2 + y^2z^2 + x^2z^2} = \sqrt{110325056695919500000000}$$

$= 332152159000$ is a square root.

Hence, $x^2y^2 + y^2z^2 + x^2z^2$ is a perfect square integer.

$\therefore x^2 + y^2 + z^2$ and $x^2y^2 + x^2z^2 + y^2z^2$ are both perfect square integers.

Thus, we have proved that the sum of the squares and the sum of the squares products are perfect square integers for $n = 4$.

$\therefore x = -521815, y = 848640, z = -302328$ are the required three integers

In general for n,

For $p = 8n(1 - n^2)$, $q = n^4 - 10n^2 + 5$, $r = n(3 + 2n^2 - n^4)$, $x = p^2 + q^2 - r^2$, $y = 2pr$, $z = 2qr$, then $x^2 + y^2 + z^2$ and $x^2y^2 + x^2z^2 + y^2z^2$ are both perfect square integers.

4. Conclusion

The main objective of this research study as stated in the objective section was deriving formulas for three integers such that the sum of their squares and the sum of products of their squares are both perfect square integers. The results and discussion section of the study visualized clearly the overall process of the derivation of the three formulas with the required conditions in descriptive process and with clearly visible practical examples so as to validate the derived formulas. Using principle of mathematical induction for depth understanding of the formulas we have checked the practicability of the derived formulas. Accordingly, the derived formulas can be seen below:

The following points are points to be improved in the future:

- As can be seen in the result and discussion section above the derivation process is very long so that after new investigation shortening the derivation process not to consume much time, resource and labor;
- Relating with practical real situations;
- The formulas become complicated as the integer n goes strictly increasing;
- The three integers themselves are not proportional;

5. Recommendation

In this research study we have derived the three required formulas and verified that for the three integers such that the sum of their squares and the sum of products of their squares are both perfect square integers. In the result and discussion section as can be clearly seen above we have shown the descriptive overall process of the derivation of the three formulas and visualized with different practical examples for verification of the result. Based on the result of this research we recommend the following:

- After new investigation of the research shortening the derivation process not to consume much time, resource and labor;
- Visualizing the pattern of the integers with practical real situations;
- Encourage other researchers to conduct research on the research area;
- The formulas become complicated as the integer n goes strictly increasing;
- The three integers themselves are not proportional;

Acknowledgement

This research study was made possible through our own funding. We are grateful to the reference books and researches. We are also grateful to the International Journal of Sciences: Basic and Applied Research (IJSBAR) for accepting the research, conducting critical peer review, giving constructive feedbacks for improvement of the research, for publishing and finally for making visible to the scientific community for open access.

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