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A new Proof of Hardy's Identity

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Abstract

We present a full proof of Hardy's identity which is of course well known of all the people interested in this field. Seemingly, there is no convenient reference, at least to our knowledge. This is what prompted us to try the proof it.

Keywords: Cauchy-Riemann equations (polar coordinates); Estimation the Integral means ; Univalent function.

1. Introduction

A function f holomorphic on the unit disk D has a Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in D.$$

It is natural to estimate the integral means of univalent function on the unit disk D by applying Hardy's identity which is one of the most useful theoretical tools in complex analysis and specifically in complex function theory as presented in Pommerenke's work [1,2].

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2. Hardy's identity

Theorem (2.1) [Hardy's identity for $I(r,\varphi)$].

Let $\Phi(t)$ be a twice continuously differentiable function, $\Psi(t) = t\frac{d}{dt}[t\Phi(t)'(t)]$, $0 \leq t < \infty$, let $\varphi(z)$ be a holomorphic in the unit disk D and

$I(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(|\varphi(re^{i\theta})|)d\theta$, $z = re^{i\theta}$, $0 \leq r < 1$,

which is the integral means of the modulus of $\varphi(z)$.

If $\varphi(z) \neq 0$ for $|z| = r$ then

$$r\frac{\partial}{\partial r}[rI'] = \frac{1}{2\pi}\int_0^{2\pi} \Psi(|\varphi(z)|)\left|\frac{z\varphi'}{\varphi}\right|^2 d\theta.$$

To this end, we will try to prove some more identities related to $|\varphi|$.

Proposition (2.1) [First identity**]**

Let $\varphi(z)$ be a holomorphic in D. If $\varphi(z) \neq 0$ in D, then

$$r\frac{\partial}{\partial r}|\varphi|(z) = |\varphi(z)|\Re\left(z\frac{\varphi'(z)}{\varphi(z)}\right). \qquad (1)$$

Proof.

Let $\varphi(z) = u(r,\theta) + iv(r,\theta)$, $z = re^{i\theta}$. Then

$$|\varphi|(z) = \sqrt{u^2 + v^2}.$$

Differentiating $\varphi(z)$ with respect to r, fixing θ then take limit along the ray where the argument is equal to θ.

$$\varphi'(re^{i\theta}) = \frac{\partial\varphi(re^{i\theta})}{\partial r} = \frac{1}{e^{i\theta}}(u_r + iv_r).$$

Now,

$$z\frac{\varphi'(z)}{\varphi(z)} = r\frac{uu_r + vv_r}{u^2 + v^2} + ir\frac{uu_r - vv_r}{u^2 + v^2},$$

which implies to

$$|\varphi(z)|\Re\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right) = r\frac{uu_r + vv_r}{\sqrt{u^2+v^2}},$$

On the other side we have

$$r\frac{\partial}{\partial r}\,|\varphi|(z) = r\frac{uu_r + vv_r}{\sqrt{u^2+v^2}}.$$

Finally, we obtain

$$r\frac{\partial}{\partial r}\,|\varphi|(z) = |\varphi(z)|\Re\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right). \blacksquare$$

Proposition (2.2) [Second identity**]**

Let $\varphi(z)$ be a holomorphic in D. If $\varphi(z) \neq 0$ in D, then

$$\frac{\partial}{\partial \theta}\,|\varphi|(z) = -|\varphi(z)|\Im\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right). \qquad (2)$$

Proof.

Let $\varphi(z) = u(r,\theta) + iv(r,\theta),\ z = re^{i\theta}$. Then

$$|\varphi|(z) = \sqrt{u^2+v^2}.$$

Differentiating $\varphi(z)$ with respect to θ, fixing r then take limit along the circle.

$$\varphi'\left(re^{i\theta}\right) = \frac{\partial\varphi\left(re^{i\theta}\right)}{\partial\theta} = \frac{1}{e^{i\theta}}(\,u_\theta + iv_\theta).$$

Now,

$$z\,\frac{\varphi'(z)}{\varphi(z)} = r\frac{uu_\theta - vv_\theta}{u^2+v^2} + i\frac{-\,uu_\theta - vv_\theta}{u^2+v^2},$$

which implies to

$$|\varphi(z)|\Im\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right) = \frac{-\,uu_\theta - vv_\theta}{\sqrt{u^2+v^2}},$$

On the other side we have

$$\frac{\partial}{\partial \theta}\ |\varphi|(z) = \frac{uu_\theta + vv_\theta}{\sqrt{u^2+v^2}}.$$

Finally, we obtain

$$\frac{\partial}{\partial \theta}\ |\varphi|(z) = -|\varphi(z)|\Im\left(z\ \frac{\varphi'(z)}{\varphi(z)}\right). \blacksquare$$

Proposition (2.3) [Third identity**]**

Let $\varphi(z)$ be a holomorphic in D. If $\varphi(z) \neq 0$ in D, then

$$r\frac{\partial}{\partial r}\ |\varphi(z)|\Re\left(z\ \frac{\varphi'(z)}{\varphi(z)}\right) - |\varphi(z)|\Im\left(z\ \frac{\varphi'(z)}{\varphi(z)}\right) = |\varphi(z)|\left|\frac{z\,\varphi'(z)}{\varphi(z)}\right|^2. \quad (3)$$

Proof. As we stated earlier

$$|\varphi(z)|\left|\frac{z\,\varphi'(z)}{\varphi(z)}\right|^2 = \frac{{u_\theta}^2 + {v_\theta}^2}{\sqrt{u^2+v^2}},$$

Here, we shall to prove that

$$r\frac{\partial}{\partial r}\ |\varphi(z)|\Re\left(z\ \frac{\varphi'(z)}{\varphi(z)}\right) - |\varphi(z)|\Im\left(z\ \frac{\varphi'(z)}{\varphi(z)}\right) = \frac{{u_\theta}^2 + {v_\theta}^2}{\sqrt{u^2+v^2}},$$

As follows

$$= \frac{\mathrm{u}^2{v_\theta}^2 + r^2u^3u_{rr} + \mathrm{u}^2{u_\theta}^2 + r^2u^2vv_{rr} + v^2\ {v_\theta}^2 + r^2\ uv^2u_{rr} + \mathrm{v}{u_\theta}^2 + r^2v^3u_{rr}}{(u^2+v^2)\sqrt{u^2+v^2}}$$
$$+ \frac{-\mathrm{u}^2{v_\theta}^2 + 2uvu_\theta v_\theta - v^2\ {u_\theta}^2 + u^3v_\theta - vu^2u_\theta + uv^2v_\theta - v^3u_\theta}{(u^2+v^2)\sqrt{u^2+v^2}}$$

Next,

$$-|\varphi(z)|\Im\left(z\ \frac{\varphi'(z)}{\varphi(z)}\right) = -\frac{\partial}{\partial \theta}\sqrt{u^2+v^2}\ \left[\frac{-\,uu_\theta - vv_\theta}{u^2+v^2}\right]\frac{\partial}{\partial \theta}\left[\frac{uu_\theta + vv_\theta}{u^2+v^2}\right],$$

So,

$$r\frac{\partial}{\partial r}\ |\varphi(z)|\Re\left(z\ \frac{\varphi'(z)}{\varphi(z)}\right) - |\varphi(z)|\Im\left(z\ \frac{\varphi'(z)}{\varphi(z)}\right) = |\varphi(z)|\left|\frac{z\,\varphi'(z)}{\varphi(z)}\right|^2 \blacksquare$$

Now, we are ready to give proof for theorem (2.1) (Hardy's identity for $I(r,\varphi)$) as follows:

Proof Theorem (2.1) (Hardy's identity for $I(r,\varphi)$).

Choose $\Phi(t) = t \Longrightarrow \Phi(t)' = 1$ such that $\Psi(t) = t\frac{d}{dt}[t] = t$.

Put $t = |\varphi(z)|$, multiplying both identities (1),(2) by $r\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$ respectively.

$$\left(r\frac{\partial}{\partial r}\right)^2 |\varphi|(z) = r\frac{\partial}{\partial r}|\varphi(z)|\Re\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right), \qquad (4)$$

$$\left(\frac{\partial}{\partial \theta}\right)^2 |\varphi|(z) = -\frac{\partial}{\partial \theta}|\varphi(z)|\Im\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right), \qquad (5)$$

Adding equation (5) to (4) in order to obtain

$$\left(r\frac{\partial}{\partial r}\right)^2 |\varphi|(z) + \left(\frac{\partial}{\partial \theta}\right)^2 |\varphi|(z) = |\varphi(z)|\Re\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right) - |\varphi(z)|\Im\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right) \quad (6)$$

Integrating (6) with respect to θ when $0 \le \theta \le 2\pi$ and using identity (3).

$$\frac{1}{2\pi}\int_0^{2\pi}\left(r\frac{\partial}{\partial r}\right)^2 |\varphi|(z)\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial}{\partial \theta}\right)^2 |\varphi|(z)d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}|\varphi(z)|\Re\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right)d\theta - \frac{1}{2\pi}\int_0^{2\pi}|\varphi(z)|\Im\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right)d\theta.$$

Finally,

$$r\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{2\pi}\int_0^{2\pi}|\varphi(z)|\,d\theta\right)\right] = \frac{1}{2\pi}\int_0^{2\pi}|\varphi(z)|\,\left|\frac{z\,\varphi'}{\varphi}\right|^2 d\theta. \blacksquare$$

In the attempt to prove Hardy identity with respect to the mean value of the modulus of φ on the circle $|z| = r$; we have been led to prove a good deal more. In particular, for the function $I_p(r,\varphi)$

$$I_p(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi}|\varphi(re^{i\theta})|^p\,d\theta, \qquad z = re^{i\theta}, \quad 0 < r < 1,$$

where p is any positive number, to this end, we will try to prove some more identities related to $|\varphi|^p$.

Theorem (2.2) [Hardy's identity for $I_p(r,\varphi)$]

Let $\Phi(t)$ be a twice continuously differentiable function, $\Psi(t) = t\frac{d}{dt}[t\Phi(t)'(t)]$, $0 \le t < \infty$, let $\varphi(z)$ be a holomorphic in the unit disk D and

$I_p(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi}\Phi(|\varphi(re^{i\theta})|^p)d\theta, \quad z = re^{i\theta}$, $0 \le r < 1,$

which is the integral means of the $|\varphi|^p(z)$,where p is a positive number.

If $\varphi(z) \neq 0$ for $|z| = r$ then

$$r\frac{\partial}{\partial r}[rI'\,] = \frac{1}{2\pi}\int_0^{2\pi}\Psi(|\varphi(z)|^{\mathrm{p}})\left|\frac{z\varphi'}{\varphi}\right|^2 d\theta.$$

To this end, we will try to prove some more identities related to $|\varphi|^p$.

Proposition (2.4) [Fourth identity**].**

Let $\varphi(z)$ be a holomorphic in D. If $\varphi(z) \neq 0$ in D, then

$$r\frac{\partial}{\partial r}|\varphi|^p(z) = p|\varphi(z)|^p\Re\left(z\frac{\varphi'(z)}{\varphi(z)}\right). \qquad (7)$$

Proof.

Let $\varphi(z) = u(r,\theta) + iv(r,\theta)$, $z = re^{i\theta}$. Then

$$|\varphi|^p(z) = (u^2+v^2)^{\frac{p}{2}}.$$

Differentiating $\varphi(z)$ with respect to r, fixing θ then take limit along the ray where the argument is equal to θ.

$$\varphi'(re^{i\theta}) = \frac{\partial\varphi(re^{i\theta})}{\partial r} = \frac{1}{e^{i\theta}}(u_r + iv_r).$$

Now,

$$z\frac{\varphi'(z)}{\varphi(z)} = r\frac{uu_r + vv_r}{u^2+v^2} + ir\frac{uu_r - vv_r}{u^2+v^2},$$

which implies to

$$p|\varphi(z)|^p\Re\left(z\frac{\varphi'(z)}{\varphi(z)}\right) = r(u^2+v^2)^{\frac{p}{2}-1}(uu_r + vv_r).$$

On the other side we have

$$r\frac{\partial}{\partial r}\,|\varphi|^p(z) = rp\,(u^2+v^2)^{\frac{p}{2}-1}\,\,(uu_r+vv_r).$$

Finally, we obtain

$$r\frac{\partial}{\partial r}\,|\varphi|^p(z) = p|\varphi(z)|^p\Re\left(\,z\,\frac{\varphi'(z)}{\varphi(z)}\right).\,\blacksquare$$

Proposition (2.5) [Fifth identity**].**

Let $\varphi(z)$ be a holomorphic in D. If $\varphi(z)\neq 0$ in D, then

$$\frac{\partial}{\partial\theta}\,|\varphi|^p(z) = -|\varphi(z)|^p\Im\left(\,z\,\frac{\varphi'(z)}{\varphi(z)}\right). \qquad (8)$$

Proof.

Let $\varphi(z)=u(r,\theta)+iv(r,\theta),\ z=re^{i\theta}$. Then

$$|\varphi|^p(z) = (u^2+v^2)^{\frac{p}{2}}.$$

Differentiating $\varphi(z)$ with respect to θ, fixing r then take limit along the circle.

$$\varphi'\big(re^{i\theta}\big) = \frac{\partial\varphi\big(re^{i\theta}\big)}{\partial\theta} = \frac{1}{e^{i\theta}}(\,u_\theta+iv_\theta).$$

Now,

$$z\,\frac{\varphi'(z)}{\varphi(z)} = r\frac{uu_\theta - vv_\theta}{u^2+v^2} + i\frac{-\,uu_\theta - vv_\theta}{u^2+v^2},$$

which implies to

$$p|\varphi(z)|^p\Im\left(\,z\,\frac{\varphi'(z)}{\varphi(z)}\right) = \,-p(u^2+v^2)^{\frac{p}{2}-1}(uu_\theta - vv_\theta),$$

On the other side we have

$$\frac{\partial}{\partial\theta}\,|\varphi|^p(z) = p(u^2+v^2)^{\frac{p}{2}-1}(uu_\theta - vv_\theta).$$

Finally, we obtain

$$\frac{\partial}{\partial\theta}\ |\varphi|^{p}(z)=-\mathrm{p}|\varphi(z)|^{p}\mathfrak{I}\left(\ z\ \frac{\varphi'(z)}{\varphi(z)}\right).\ \blacksquare$$

Proposition (2.5) [Sixth identity**].**

Let $\varphi(z)$ be a holomorphic in D. If $\varphi(z)\neq 0$ in D, then

$$rp\frac{\partial}{\partial r}\ |\varphi(z)|^{p}\mathfrak{R}\left(\ z\ \frac{\varphi'(z)}{\varphi(z)}\right)-\mathrm{p}|\varphi(z)|^{p}\mathfrak{I}\left(\ z\ \frac{\varphi'(z)}{\varphi(z)}\right)=p^{2}|\varphi(z)|^{p}\left|\frac{z\,\varphi'(z)}{\varphi(z)}\right|^{2}. \quad (9)$$

Proof. As we stated earlier

$$p^{2}|\varphi(z)|^{p}\left|\frac{z\,\varphi'(z)}{\varphi(z)}\right|^{2}=\ p^{2}(u^{2}+v^{2})^{\frac{p}{2}-1}\,(\,u_{\theta}{}^{2}+v_{\theta}{}^{2}).$$

Here, we shall to prove that

$$rp\frac{\partial}{\partial r}\ |\varphi(z)|^{p}\mathfrak{R}\left(\ z\ \frac{\varphi'(z)}{\varphi(z)}\right)-\mathrm{p}|\varphi(z)|^{p}\mathfrak{I}\left(\ z\ \frac{\varphi'(z)}{\varphi(z)}\right)=p^{2}(u^{2}+v^{2})^{\frac{p}{2}-1}\,(\,u_{\theta}{}^{2}+v_{\theta}{}^{2}),$$

As follows

$$rp\frac{\partial}{\partial r}\ |\varphi(z)|^{p}\mathfrak{R}\left(\ z\ \frac{\varphi'(z)}{\varphi(z)}\right)=rp\left[\frac{\partial}{\partial r}r(u^{2}+v^{2})^{\frac{p}{2}-1}\,(uu_{r}+vv_{r})\right]$$

$$=r^{2}p(u^{2}+v^{2})^{\frac{p}{2}-1}(u_{r}{}^{2}+uu_{rr}+v_{r}{}^{2}+vv_{rr})+rp(uu_{r}+vv_{r})(u^{2}+v^{2})^{\frac{p}{2}-1}$$
$$+\,r^{2}p\left(\frac{p-2}{2}\right)(uu_{r}+vv_{r})(u^{2}+v^{2})^{\frac{p}{2}-2}(2uu_{r}+2vv_{r}).$$

Now, we have to be careful in order to make a little bit change between each of

$u(r,\theta),v(r,\theta)$ through applying Cauchy -Riemann equations (in polar coordinats)

$u_{r}=\frac{1}{r}v_{\theta}\ ,v_{r}=-\frac{1}{r}u_{\theta},$

as follows

$$rp\frac{\partial}{\partial r}\ |\varphi(z)|^{p}\mathfrak{R}\left(\ z\ \frac{\varphi'(z)}{\varphi(z)}\right)$$
$$=p(u^{2}+v^{2})^{\frac{p}{2}-1}(v_{\theta}{}^{2}+ruv_{r\theta}-\mathrm{uv}_{\theta}+\mathrm{u}_{\theta}{}^{2}\ -\mathrm{rvu}_{r\theta}\ +\mathrm{vu}_{\theta})+p(uv_{\theta}-vu_{\theta})(u^{2}+v^{2})^{\frac{p}{2}-1}$$
$$+\,p(p-2)(u^{2}+v^{2})^{\frac{p}{2}-2}(uv_{\theta}-vu_{\theta})^{2}.$$

Similarly, we can calculate $-|\varphi(z)|\Im\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right)$ as well

$$-p|\varphi(z)|\Im\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right)$$
$$= p(u^2+v^2)^{\frac{p}{2}-1}({u_\theta}^2+{v_\theta}^2) + p^2(uu_\theta+vv_\theta)^2(u^2+v^2)^{\frac{p}{2}-2}$$
$$-\,2p(uu_\theta+vv_\theta)^2(u^2+v^2)^{\frac{p}{2}-2}.$$

So,

$$rp\frac{\partial}{\partial r}\,|\varphi(z)|^p\Re\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right) - p|\varphi(z)|\Im\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right) = p^2(u^2+v^2)^{\frac{p}{2}-1}({v_\theta}^2+{\mathrm{u}_\theta}^2\,)$$

$$,\ rp\frac{\partial}{\partial r}\,|\varphi(z)|^p\Re\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right) - p|\varphi(z)|\Im\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right) = p^2|\varphi(z)|^p\left|\frac{z\,\varphi'(z)}{\varphi(z)}\right|^2 \blacksquare$$

Proof Theorem (2.2) (Hardy's identity for $I_p(r,\varphi)$).

Choose $\Phi(t) = t^P \Longrightarrow \Phi(t)' = pt^{p-1}$ such that $\Psi(t) = t\frac{d}{dt}[pt^p] = p^2t^p$.

Put $t^p = |\varphi(z)|^p$, multiplying both identities (7),(8) by $r\frac{\partial}{\partial r},\frac{\partial}{\partial \theta}$ respectively.

$$\left(r\frac{\partial}{\partial r}\right)^2 |\varphi|^p(z) = pr\frac{\partial}{\partial r}|\varphi(z)|^p\Re\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right), \tag{10}$$

$$\left(\frac{\partial}{\partial \theta}\right)^2 |\varphi|^p(z) = -\mathrm{p}\frac{\partial}{\partial \theta}|\varphi(z)|^p\Im\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right), \tag{11}$$

Adding equation (11) to (10) in order to obtain

$$\left(r\frac{\partial}{\partial r}\right)^2 |\varphi|^p(z) + \left(\frac{\partial}{\partial \theta}\right)^2 |\varphi|^p(z) = pr\frac{\partial}{\partial r}|\varphi(z)|^p\Re\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right) - \mathrm{p}\frac{\partial}{\partial \theta}|\varphi(z)|^p\Im\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right) \tag{12}$$

Integrating (12) with respect to θ when $0 \le \theta \le 2\pi$ and using identity (9).

$$\frac{1}{2\pi}\int_0^{2\pi}\left(r\frac{\partial}{\partial r}\right)^2 |\varphi|^p(z)\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial}{\partial \theta}\right)^2 |\varphi|^p(z)d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}|\varphi(z)|^p\Re\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right)d\theta - \frac{1}{2\pi}\int_0^{2\pi}|\varphi(z)|^p\Im\left(z\,\frac{\varphi'(z)}{\varphi(z)}\right)d\theta.$$

Finally,

$$r\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{2\pi}\int_0^{2\pi}|\varphi(z)|^p\,d\theta\right)\right] = \frac{p^2}{2\pi}\int_0^{2\pi}|\varphi(z)|^p\,\left|\frac{z\,\varphi'}{\varphi}\right|^2 d\theta.\blacksquare$$

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