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Phase Transition of Geographic Networks: Monte Carlo Tests of Madagascar's Road Network

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Abstract

A geographic network is a particular case of random network in which links among nodes are under geographic constraints; node positions are frozen in the space and the space itself is bounded by a real geographic border. This paper studies the phase transition of these networks for Madagascar's road network using the random network model. The Monte Carlo method is used to study the site percolation of the network on the projected map of the country. The Gilbert disc model gives us the connection threshold radius for the network and the effect of the geographical connection range on the phase transition. We also note that for a certain range, the radius has a noticeable effect on the birth process of clusters that takes place inside the network. The study ends by presenting a generic example of a propagation process inside the network and superimposed on a map.

Keywords: Geographic network; Phase transition.

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1. Introduction

Due to the rapid development of global telecommunication networks and the increasing access to computing power by the public, complex network theory has evolved to a mature development phase and its application covers several disciplines, such as brain cell networks, social interactions, and telecommunication networks. The particular case of geographic networks has also attracted interest from the research community to obtain a deeper understanding of issues such as urban street patterns, road classification, and network robustness.

A geographic network belongs to the general area of spatial networks. A spatial network is a particular kind of network that is embedded in space. Nodes occupy precise positions in the two- or three-dimensional Euclidean space and edges represent physical connections or interactions [1,2,3] Spatial networks owe their popularity due to the fact that they allow combining the metric properties of the network with graph theory. Thus, spatial networks have been applied to neuronal networks in culture [4], disease spread, traffic congestion, and city structure. Latorab and his colleagues [2] dedicated a study to spatial networks in Euclidean space while Massimo and Ronald [5] explored the phase transition of a random network in an infinite plane. Barthelemy [3] provided an extensive result about structure of general spatial networks.

On the topic of geographical networks, the work of Miwa and his colleagues [6] on non-growing geographical networks has given a first inspiration. We can also cite recent studies by Lu and Duan [7] about the robustness of urban road networks, taking into account the connection among urban communities, with extensive data from some of the world's largest cities: London, Paris, and Beijing. Gao and his colleagues [8], discussed the robustness of geographical networks by qualifying an attack as distributed or concentrated according to its area.

The study of percolation emerged from the field of physics in the 1940s [9]. The problem formulation given by Broadbent and Hammersley in 1957 gives us a clear understanding of the percolation model; "Suppose we immerse a large porous stone in a bucket of water. What is the probability that the center of the stone is wetted?" Figure 1 illustrates the problem in a two-dimensional space in which small pores inside the stone act as nodes and the small tunnel connecting them as edges. Percolation theory studies the birth process of networks inside a given system [10].

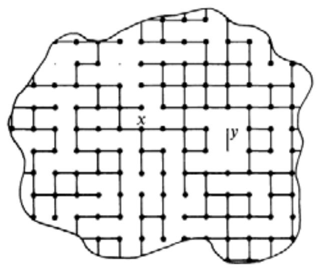

Figure 1: 2D formulation of the percolation problem.

Finally, a phase transition is known as the phenomenon under which a system has an abrupt change in its global behavior due to a small change of parameters [5]. In physics, phase transition is the study of system under critical state, such as sublimation or evaporation. For a complex network, phase transition can be observed when links are gradually added to a random graph. After a certain number of added links, the threshold (the dislocated random graph) is starting to host families of connected nodes until a complete graph is derived.

Several studies have focused on finding the percolation threshold. Reference [11] reviews the advances made in this area and, at the same time, establishes standard models to deal with geometric random graphs: the Gilbert or disc model, the k-nearest neighbor model, and the Voronoi model. Besides the model of percolation on a continuum space developed by Yogeshwaran and Iyer and his colleagues [12], we can also cite the model developed by Muller and Rudolph-Lilith [13] to compute the percolation threshold of random graphs using minimal Hamiltonian cycles. Moore and Mertens [14] enquired about the percolation threshold in two dimensional continuum space using the union-find algorithm. The statistical properties of a geometric random graph in a plane and the percolation threshold have been studied by Melchert in [15]. In [16], Saniee and Bradonjic developed a study of bootstrap percolation on random geometric graphs that can be applied to wireless connections and contagion in a geographic space. In [17], Davis got deeper into the percolation of a spatial network and disease spread. For geographic networks, Yang and his colleagues [18] developed a "hollow model for immunization" from such networks. In [19], an enquiry about the percolation threshold brings more precision for the case of fragile networks. Finally, we cite the inspiring work of Gandino and Ferrero [20] concerning the degree distribution of unit disc graph with uniformly distributed nodes on a rectangular surface, with a geometric analysis of the border effect.

For this study, we concentrate on the road transportation network of Madagascar. For this purpose, the rest of the paper is divided into three parts. The first part relates to the methodology used for building the random network model and the branching method kept for the study of the phase transition. This part will be followed by the results and a final concluding part, with potential extensions for further enquiry.

2. Methodology

The complex network model is built from the official road map of Madagascar issued by the national geographic center (FTM). From the current existing configuration, the final graph model contains 171 nodes, which represent the 171 cities and road connections on the island and 251 edges, representing the 251 different routes. Figure 2 illustrates the positioning of the nodes on a map of the country (left) and a simplified representation of the 251 current connections (right). The x and y axes correspond to longitude and latitude, respectively.

We are interested in the transition phase of the geographical random graph obtained.

The classic model of the random graph allows the formation of connections over a long distance, which also allows the formation of unbounded clusters inside the network. This approach is not realistic in many cases of real-life networks. Indeed, in the case of information spread, goods transfer, and pathogenic spread, agents have to travel across the network from location to location, which disallows the creation of a random long-range connection. For example, a direct connection between nodes located in the north and south regions of the map is allowed by random creation, but are unrealistic from a transportation viewpoint.

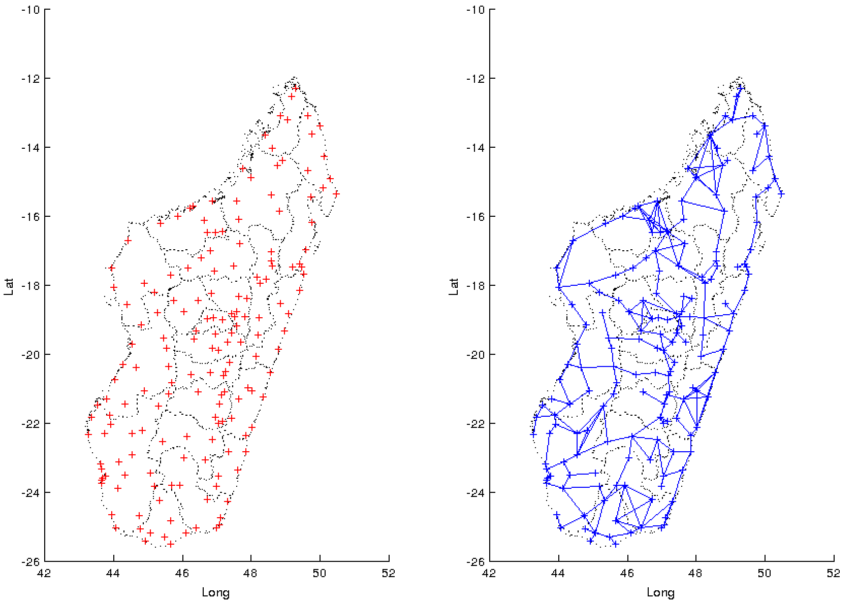

Figure 2: Network geographical configuration.

The diffusion process is more likely to spread from a central point and spread inside the network. The network that takes birth from a diffusion process tends to be connected to the same giant component, disallowing the formation of small and separated clusters inside the network. Therefore, a geometric rule has to be added to the connection creation law of the random graph [3,21].

Similar to radio transmission [22,23], the disc model is used to study the phase transition of the network. The evolution of the system is measured according to the size of the largest cluster inside the network [24].

The disc model used in information technology happens to be more realistic and, consequently, is used to model the branching process. Close to the WiFi model of [22] and [23], the principle of the disc model is illustrated in the Figure 3. The connection between two nodes is possible only if the Euclidean distance between the two nodes is less that the connection range r.

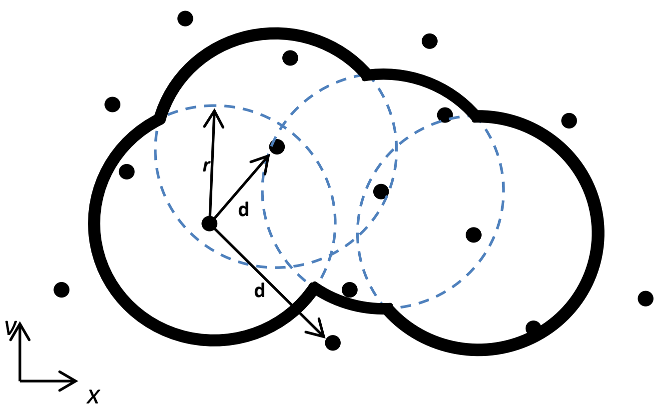

Figure 3: Giant component formation.

For this case, the geometry constraint [3] for the network is:

$$M(i,j)=\begin{cases} x=1 & if \;\; d(n_i,n_j)\le r \\ x=0 & if \;\; d(n_i,n_j)> r \end{cases} \quad (1)$$

The network is built by joining the disc model with the properties of a classic geometric graph.

For a planar geometric graph with uniformly randomly distributed nodes, the degree distribution follows the binomial distribution as follows [1,25,26]:

$$P(k)=\binom{n-1}{k}p^k(1-p)^{n-k-1} \quad (2)$$

For a model where nodes are randomly placed in the plane according to a Poisson distribution, the density is given by $\frac{N}{A}$. Since the neighbor node is contained inside the area πr^2, the average node degree equals to $<k>=\frac{N\pi r^2}{A}$ and the probability distribution becomes, respectively, in binomial and in the approximated Poisson form [3,20,26]:

$$P(k)=\binom{n-1}{k}\left(\frac{N\pi r^2}{A}\right)^k\left(1-\frac{N\pi r^2}{A}\right)^{n-k-1} \quad (3)$$

$$P(k)=\frac{\left(\frac{N\pi r^2}{A}\right)^k}{k!}e^{\frac{N\pi r^2}{A}} \quad (4)$$

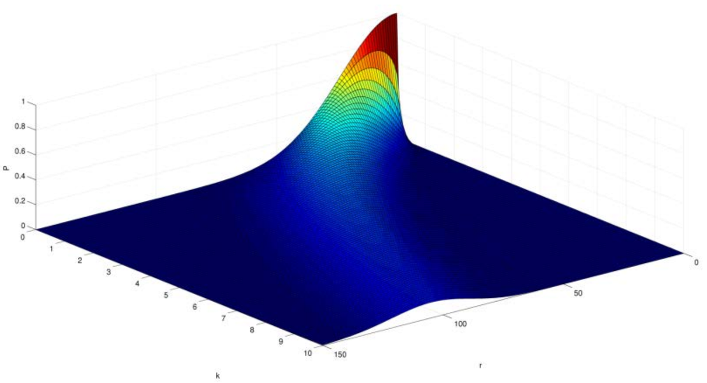

Figure 4: Probability distribution P versus degree *k* (0–10), radius *r* (0–150 km).

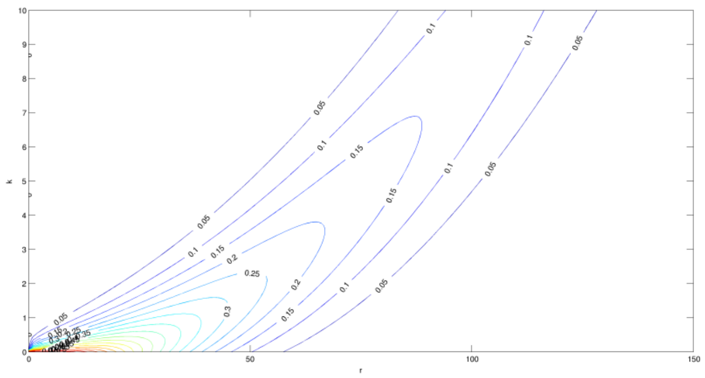

Figure 5: Contour plot of the degree distribution.

The last two figures show the distribution of the probability density for various values of the radius range r and the node degree taken from equation (4). Note that for the real configuration, the average node degree $\langle k \rangle = 2.93$. The results in the following part relate the Monte Carlo [24] result for the threshold radius of the model.

To allow any geometric computation on the map, a projection into the Laborde Coordinate System, proper for the building of the map of Madagascar, has been implemented [27]. The transformation from a spherical coordinate system (latitude, longitude) to a Cartesian grid (x, y) enables computation of the distance between each couple of nodes for the branching process. Digital maps have been processed with Qgis.

3. Results

The model presented as Figure 2 (left) comprises 171 nodes spread across an area of 594,856 km2. The boundary of the two dimensional space is the border of the country and nodes have fixed locations on the surface according to their GPS coordinates.

For the current configuration, the metric of the random graph model shown in the Figure 2 (right) is quickly summarized by the boxplot below. The maximum and minimum distances between nodes on the map are, respectively, 345 and 7 km; the 25 percentile is 58 km; and the 75 percentile equals 135 km. The median of the distances is 89 km and the average route length is 100 km.

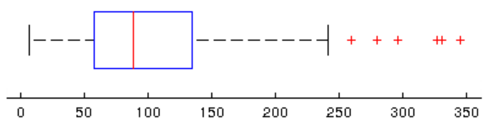

Figure 6: Metric of routes inside the current network.

Concerning the phase transition, Figure 7 shows three results for the range radius r. As expected, a small value of $r < 50$ km disallows any connection to take place inside the network; this is the case for $r = 50$ km. For 50 km $< r \leq 100$ km, links start to appear inside the network but the forming connections are confined among geographically close nodes, disallowing the possibility to create stable connected components. For a research range greater than 100 km, clusters start to birth inside the network.

On Figure 8, cases for r > 100 km are under scrutiny. Figure 8 shows the evolution of the phase transition curve according to the geographical parameter r. For a value of r close to the connection threshold, clusters take place and, gradually, reach the maximum size of the giant component, as more sites are added. The phase transition process gets faster as r increases, to finally reach the shape of the classic random graph model. We can consider that for a value of r above 300 km, all nodes are reachable from any randomly chosen starting node. The model thus converges to a classic random graph, and the range r has no further effect on the formation of a giant component. The x and y axes correspond, respectively, to the fraction f of nodes added to the network and the relative size of the giant component.

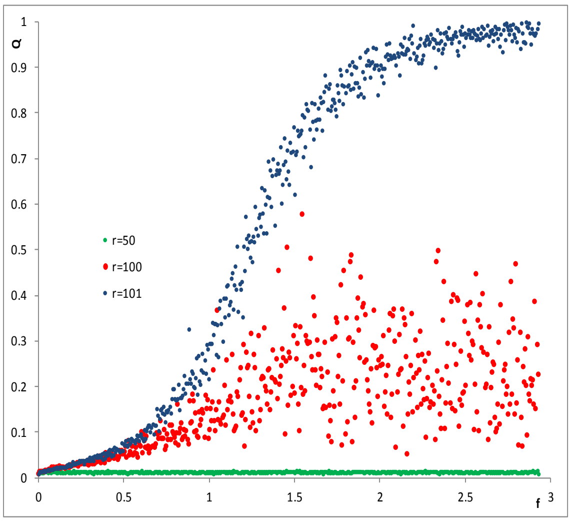

Figure 7: Size of the giant component for *r* = 50 km, 100 km, and 101 km.

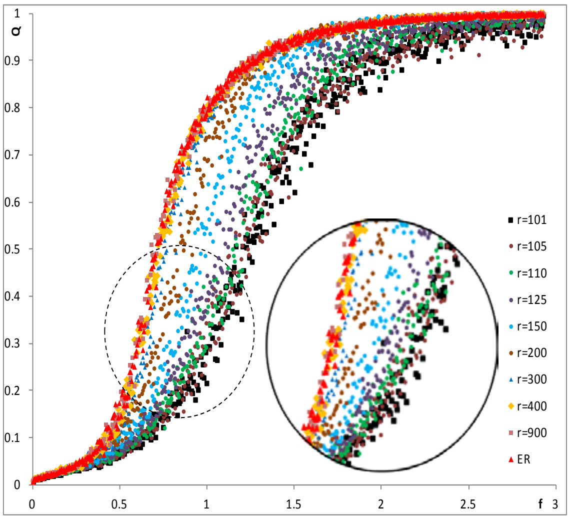

Figure 8: Evolution of the phase transition for *r* above 101 km.

To get into the detail of the results in Figure 8, the two following figures capture the size of the giant component at the percolation threshold [25] f = 0.34 (Figure 9) and f = 1 (Figure 10). Figure 10 shows clearly that for r > 200 km and based on the size of the giant component, the network behaves similarly to a random graph.

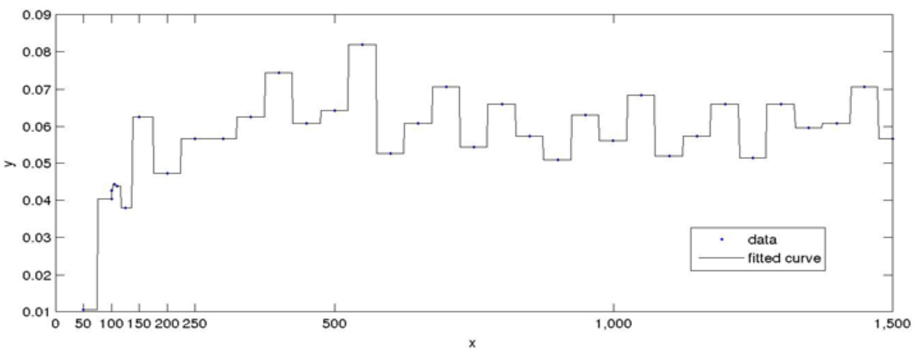

Figure 9: Size of the giant component (*y*) around the percolation threshold versus radius (*x*).

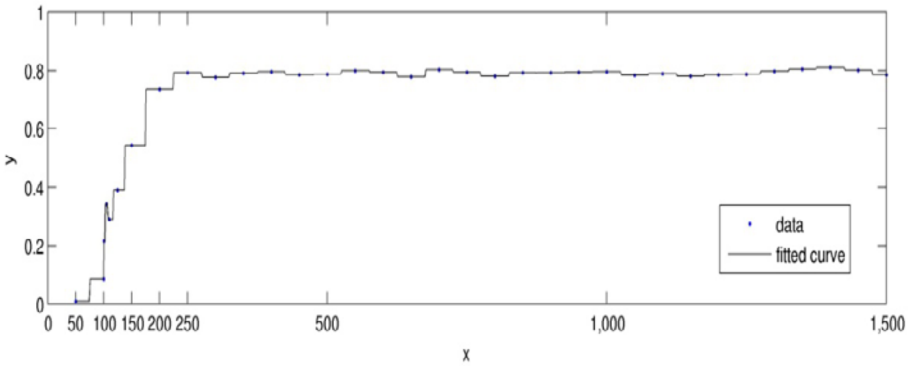

Figure 10: Size of the giant component (*y*) at full connection.

Figure 11 illustrates a sample result of the birth process of the connected components inside the network. The propagation starts from the node corresponding to the capital city of Antananarivo, then spreads according to the branching process of a random geographical network: r = 101 km.

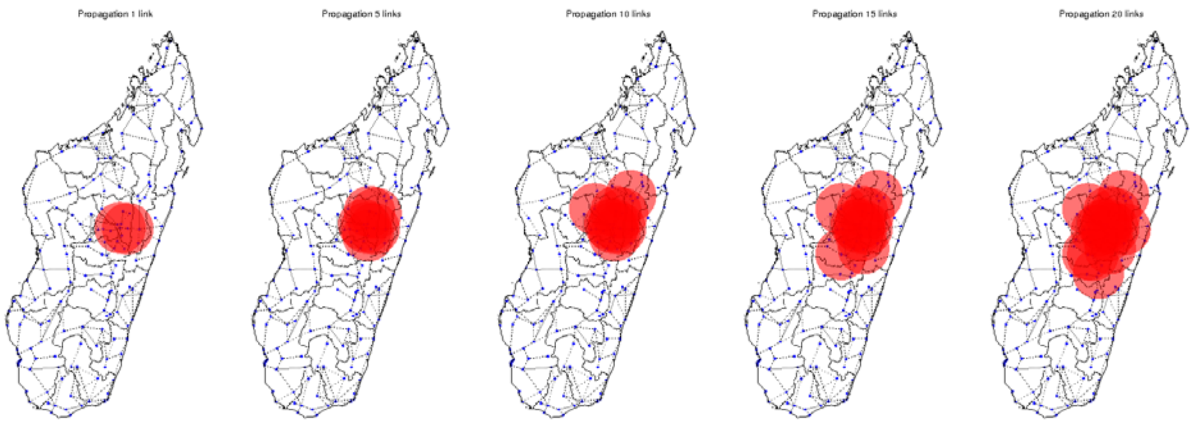

Figure 11: Sequence of diffusion with radius *r* = 101 km. Diffusion center: the nation's capital city. Screenshots for 1st, 5th, 10th, 15th, and 20th link created.

4. Conclusion and Outlook

This study examines the road transportation network of Madagascar and allows us to revisit important aspects of complex network models as applied to geographic networks. A geographic network is particular to the extent that links are geographically constrained, nodes are spread out in space with fixed coordinates, and the space boundary is complex, without mathematical formulation. The random network model helps us understand the transportation system and the geographic constraint is modeled with the disc branching process. Monte Carlo simulation on the phase transition helps discover the threshold radius for connection and propagation to take place inside the network, and its behavior under critical phase. This study supports a fundamental comprehension regarding the inherent properties of the country's road network. The generic model is flexible enough to permit additional studies, such as on spread of information, epidemics, or any phenomena that can diffuse across a network.

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