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Heat Transfer Equation With Delay for Media With Thermal Memory

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Abstract

A new model for heat transfer in this paper is proposed. It combines idea of medium with memory and phase-lag model. Equation for a temperature field based on new heat transfer model was obtained and investigated with wave-like solutions. New model was compared with common models for non-stationary heat transfer by its wave-like solutions amplitude attenuation, wave length and phase velocity. It was shown that model with memory is equivalent to a hyperbolic model of heat transfer. While new combined model is equivalent with a phase-lag model for a low frequencies but differs for a high frequencies. Both this models predict possibility of undamped thermal waves, but phase-lag model predict a numerous quantity of undumped thermal waves, while combined model predict undumped wave for a one frequency.

Keywords: heat transfer; medium with thermal memory; undumped thermal waves; thermal relaxation time

1. Introduction

For a long period of time classical heat transfer model was providing satisfying results, in description of well know at that time heat transfer processes. But in a middle of twenty century the first step was made to build a heat transfer model for a case of extremely non-stationary situation. Heat transfer equation for this case is named Maxwell-Cattaneo equation [1]:

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$$\overline{q}(\overline{r},t)+\tau\frac{\partial\overline{q}(\overline{r},t)}{\partial t}=-\lambda\cdot gradT\left(\overline{r},t\right) \quad (0.1)$$

But in the last decade of the twenty century and in the first decade investigations in the field of non-stationary heat transfer processes became really widespread. The reason for it is a development of technologies. Problems of a short laser impulse beam interaction with materials (welding, surgery) [1], thermal processes in quasi equilibrium plasma [2], thermal wave microscopy [3] or even thermal effects in modern high frequency electronic microchips [4] demands new knowledge about fast heat transfer processes. In the literature of this period models of heat transfer with phase lag proposed by Tzou were widely discussed [5,6,7,8]. Different variations of models with phase lag are out of the field of interest in this paper; here we focus only on one of them. Double phase lag model (DPL):

$$q\left(\overline{r},t+\tau_q\right)=-\lambda\cdot gradT\left(\overline{r},t+\tau_T\right) \quad (0.2)$$

Or simpler model - single phase lag (SPL) heat transfer equation:

$$q\left(\overline{r},t+\tau_q\right)=-\lambda\cdot gradT\left(\overline{r},t\right) \quad (0.3)$$

From (1.3) we can easily obtain (1.1) as a first order approximation. Relations between heat flux and temperature gradient(1.1), (1.3) gives the following equations for temperature field:

$$\rho c\frac{\partial T\left(\overline{r},t\right)}{\partial t}+\tau\rho c\frac{\partial^2 T(\overline{r},t)}{\partial t^2}=div\left[\lambda\cdot gradT\left(\overline{r},t\right)\right], \quad (0.4)$$

and:

$$\rho c\frac{\partial T\left(\overline{r},t+\tau\right)}{\partial t}=div\left[\lambda\cdot gradT\left(\overline{r},t\right)\right], \quad (0.5)$$

respectively. The physical meaning of a time delay parameter is time of thermal relaxation to equilibrium state [9,10]. Another modification of heat transfer equation is a model of medium with memory [4]:

$$\overline{q}\left(\overline{r},t\right)=-\int_{-\infty}^{t}\frac{1}{\tau}\cdot e^{-\frac{t-t'}{\tau}}\cdot\lambda\cdot grad\left(T\left(\overline{r},t'\right)\right)dt' \quad (0.6)$$

Or corresponding equation for temperature field is:

$$\rho c\frac{\partial T\left(\overline{r},t\right)}{\partial t}=\int_{-\infty}^{t}\frac{1}{\tau}\cdot e^{-\frac{t-t'}{\tau}}div\left[\lambda\cdot gradT\left(\overline{r},t'\right)\right]dt' \quad (0.7)$$

In the present paper, we report on new model of heat transfer that combines aspects of medium with memory and thermal relaxation heat transfer and compare heat transfer models with non-stationary periodical solutions.

2.Materials and methods

In order to modify heat transfer models (1.5) and (1.7) we need to notice that both models can be written in a general form:

$$\rho c\frac{\partial T\left(\overline{r},t\right)}{\partial t}=\int_{-\infty}^{t}G\left(t,t'\right)div\left[\lambda\cdot gradT\left(\overline{r},t'\right)\right]dt' \quad (2.1)$$

For SPL heat transfer equation:

$$G\left(t,t'\right)=\delta\left(t-\tau-t'\right), \quad (2.2)$$

In (2.2) $\delta\left(t-\tau-t'\right)$ is a delta function. For model with thermal memory:

$$G(t,t') = \frac{1}{\tau} \cdot e^{-\frac{t-t'}{\tau}} \tag{2.3}$$

Here we suggest a kernel that combines properties of (2.2) and (2.3)

$$G(t,t') = \frac{\tau_1 + \tau_2}{\tau_1^2} \cdot e^{-\frac{t-t'}{\tau_1}} \cdot \left(1 - e^{-\frac{t-t'}{\tau_2}}\right) \tag{2.4}$$

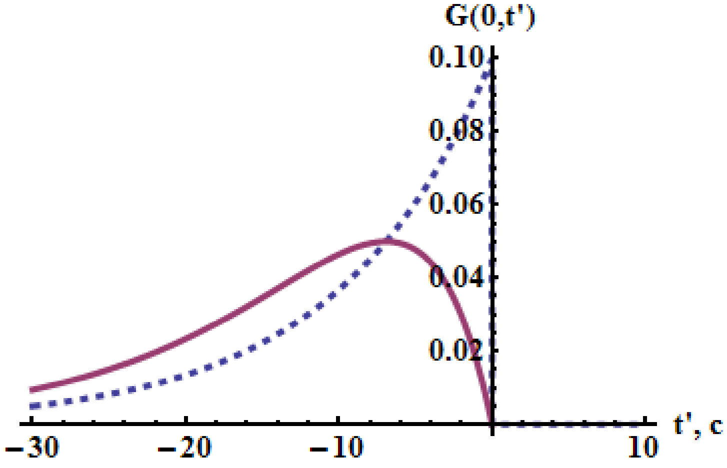

Fig. 1. Kernel functions for medium with memory and combined model.

On fig. 1 kernel functions (2.3) and (2.4) are plotted with condition t=0. Material parameters τ, τ_1, τ_2 were taken all the same and equal to 10 seconds. Usually this value of thermal relaxation parameters is used in problems concerned with bio tissue. Combined kernel function has a local maximum at $t' = -\tau_2 \cdot ln\left(1 + \frac{\tau_1}{\tau_2}\right)$; it is similar with kernel function for a SPL model. But also combined kernel function is averages values of temperature gradient in all previous moments; it is similar with memory model. Here we should mention that it is not necessary for both parameters τ_1, τ_2 in combined kernel to have the same values. Corresponding equation for temperature field will be:

$$\rho c \frac{\partial T(\bar{r},t)}{\partial t} = \int_{-\infty}^{t} \frac{\tau_1 + \tau_2}{\tau_1^2} \cdot e^{-\frac{t-t'}{\tau_1}} \cdot \left(1 - e^{-\frac{t-t'}{\tau_2}}\right) div\left[\lambda \cdot gradT(\bar{r},t')\right] dt' \tag{2.5}$$

The next step is to distinguish these models via their non-stationary solutions. For this purpose wave-like solutions will be used

$$T(x,t) = T_0 \cdot e^{i(\omega t - kx)} \tag{2.6}$$

In what follows we will regard only one-dimensional models. Substituting (2.6) to one-dimensional forms of equations for temperature field (1.4), (1.5), (1.7) and (2.5) we will obtain dispersion relations:

$$k_H = \sqrt{\frac{\rho c \omega}{\lambda} \sqrt{\tau^2 \omega^2 + 1}} \cdot Exp\left(-i \cdot 0.5 \cdot arctg\left(1/_{\tau\omega}\right)\right) \tag{2.7}$$

$$k_{SPL} = \sqrt{\frac{\rho c \omega}{\lambda}} \cdot Exp\left(i\left(\frac{\omega\tau}{2} - \frac{\pi}{4}\right)\right) \tag{2.8}$$

$$k_M = \sqrt{\frac{\rho c \omega}{\lambda}\sqrt{\tau^2\omega^2+1}} \cdot Exp\left(-i \cdot 0.5 \cdot arctg\left(1/\tau\omega\right)\right) \tag{2.9}$$

$$k_C = \sqrt{\frac{\rho c \omega^2}{\lambda}\frac{\tau_1^2}{\tau_1+\tau_2}\left(\tau_1^2+2\tau_1\tau_2\right)\sqrt{1+tg^2\varphi}} \cdot Exp\left(-i \cdot \varphi\right)$$
$$\varphi = arctg\frac{\tau_1+\tau_2-\omega^2\tau_1^2\tau_2}{\omega\tau_1^2+2\omega\tau_1\tau_2} \tag{2.10}$$

Physical meaning of wave number imaginary part is attenuation of temperature wave, while real part is a phase number. As it appears, wave number for a hyperbolic heat transfer model is equivalent with wave number for a model with thermal memory. It means that any non-stationary solution will be common for these two models.

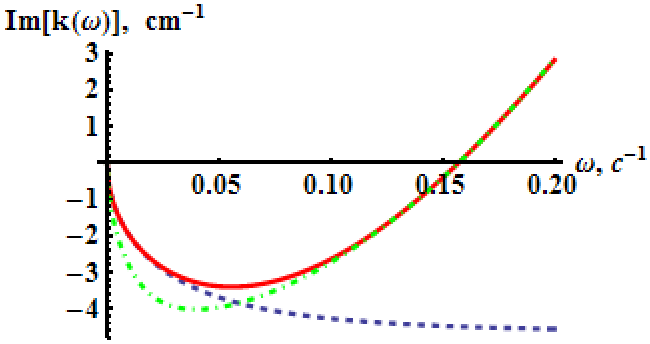

Fig. 2. Attenuation coefficients for SPL model, model with memory and combined model.

On figure 2 attenuation coefficients for SPL model (dashed curve), model with memory (continuous curve) and combined model (dot dashed curve) are plotted. First of all we should make some remarks about material parameters that were used. Temperature conductivity and time of thermal relaxation were taken as common values for bio tissue $\frac{\lambda}{\rho c} = 1.14 \cdot 10^{-3}\frac{cm^2}{s}, \tau = 10\,s$. But for combined model it is not so simple to choose material parameters, temperature conductivity we will chose the same but new model is using two time parameters. As we can see on fig. 2 both SPL model and combine model predict undumped temperature waves. The frequency of undumped waves should be the same in both models for bio tissue; it will be guaranteed if the next relation is true:

$$\tau_2 = \tau_1 \cdot \left(\frac{\pi^2}{4}\frac{\tau_1^2}{\tau^2}-1\right)^{-1} \tag{2.11}$$

In order to make attenuation in combined model of the same order as in SPL model and model with memory τ_1 was chosen equal to 1 second and $\tau_2 = -1.0253$. It may seem strange for time parameter to be negative. But we can regard it like a change of signs in (2.4):

$$G(t,t') = \frac{\tau_1-\tau_3}{\tau_1^2} \cdot e^{-\frac{t-t'}{\tau_1}} \cdot \left(1-e^{\frac{t-t'}{\tau_3}}\right) \tag{2.12}$$

Where $\tau_3 = |\tau_2|$. Kernel function is positive for these values of parameters at any moment of time. In this case combined model not only brings the same value of undumped waves but also similar attenuation parameters. On fig. 3 wavelength as function of frequency is plotted. On fig. 4 phase velocity for the same parameters is plotted. There

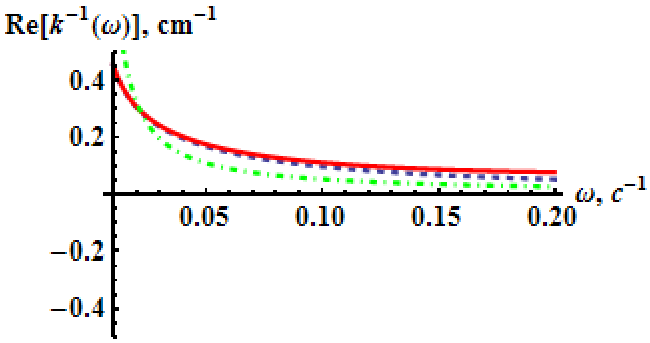

Fig. 3.Wave length for SPL model, model with memory and combined model.

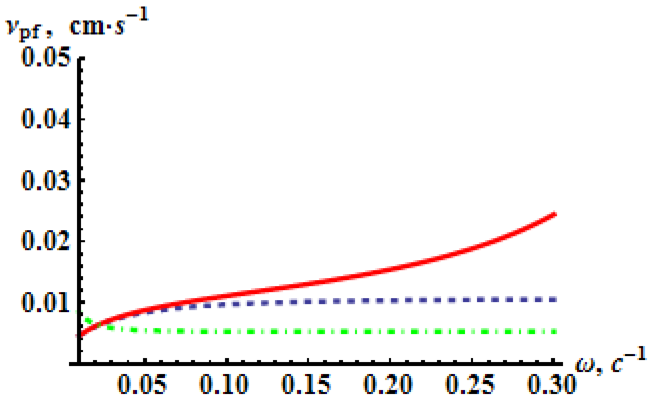

Fig. 4. Phase velocity for SPL model, model with memory and combined model.

is no principal difference in wavelength but phase velocity according to SPL model increasing with frequency while in combine model it almost stays the same accept small decrease at very small frequencies. This information can be used for example in order to distinguish these two models in experiments. In presented above results time parameters for combined model were chosen this way that provides similar attenuation for temperature waves, but in what follows we show that attenuation coefficient in combine model more differ a lot.

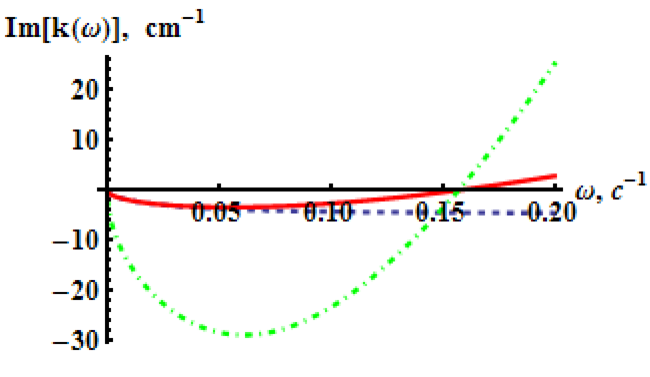

Fig. 5. Attenuation coefficients for SPL model, model with memory and combined model $\tau_1 = 5s$.

On fig. 5 attenuation coefficient is plotted supposing $\tau_1 = 5s$ and on fig. 6 value $\tau_1 = 0.5s$ were used. As it can be seeing attenuation in combine model can be much bigger and much smaller than that in SPL model and model with memory. Proposed model cover much more different situations, this is advantage comparing with SPL model, but there is also one aspect of problem that were not discussed yet. It is undumped waves.

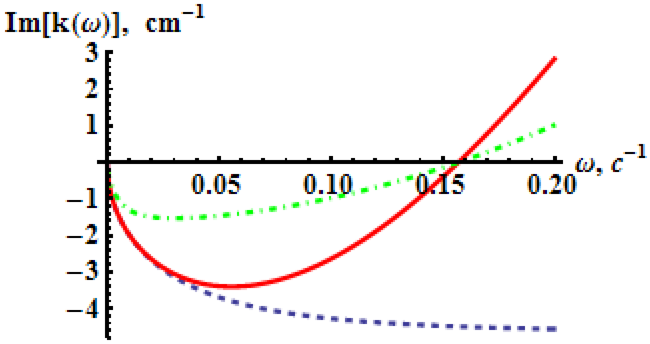

Fig. 6. Attenuation coefficients for SPL model, model with memory and combined model $\tau_1 = 0.5s$.

3.Results and discussions

As well both SPL model and combined model predicts existence of undumped temperature waves. But in combined model attenuation coefficient equal to zero only for a one frequency and for higher frequencies harmonic solutions became unstable (waves with increasing amplitudes). These solutions are prohibited. So we can regard processes for a frequencies $\omega < \omega_0$ where ω_0 states for undumped waves. SPL model predict not only one undumped harmonic but countable set of them. But in SPL model undumped harmonics separated with continuous intervals of prohibited solutions, that is why proposed new model seems to be more "physical", in it only two continuous intervals exists. First interval is an interval of allowed solutions and another one of prohibited solutions. It makes a sense. Because non-stationary models break assumption of local stationary that is one of backgrounds to classical thermodynamics. In this assumption all macro subsystems reach equilibrium states much faster than macro-parameters changes, but in non-stationary models the goal is to describe situations when characteristic time for changes in macro-parameters is comparable with a time needed for macro subsystems to reach equilibrium state. So situation when macro parameters are changing much faster than subsystems are relaxing to equilibrium states is impossible that is what combine model is stating. As it can be seen combined model and SPL model can't be distinguished with measuring frequency of undumped wave so it is also needed to measure such parameter as wavelength. Undumped waves can be very useful in many applications. Non-stationary heat transfer regimes are used in materials structure diagnostics. But also we may suppose that undumped thermal waves can be useful also to transmit signals through plasma, where attenuation of electromagnetic waves is very high and in purpose to design heating conduits with extremely low unwanted thermal energy loss.

In this paper new model for non-stationary regimes of heat transfer were proposed. It was investigated with wavelike harmonic solutions together with SPL model, hyperbolic, and medium with memory models. It was shown that both SPL and combined model predicts existence of undumped thermal waves, while model with memory and hyperbolic model are equivalent in terms of harmonic solutions and do not have undumped wavelike solutions. It makes important further studding of non-stationary heat transfer regimes, what can be useful in thermal wave materials structure diagnostics [11] or in cooling processes of modern high frequency microchips.

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