On a Generalized $H^h$- Birecurrent Finsler Space

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Abstract

In the present paper, a Finsler space whose curvature tensor $H^i_{jkh}$ satisfies $H^i_{jkh\ell \ell m} = a_{\ell \ell m}H^i_{jkh} + b_{\ell \ell m}(\delta^i_{\ell \ell}g_{jk} - \delta^i_{\ell \ell}g_{jk})$, $H^i_{jkh} \neq 0$, where $a_{\ell \ell m}$ and $b_{\ell \ell m}$ are non-zero covariant tensor fields of second order called recurrence tensor fields, is introduced, such space is called as a generalized $H^h$—birecurrent Finsler space. The associate tensor $H^i_{fjk}$ of Berwald curvature tensor $H^i_{jkh}$, the torsion tensor $H^i_{jk}$, the deviation tensor $H^i_{j}$, the Ricci tensor $H_{jk}$, the vector $H_k$ and the scalar curvature $H$ of such space are non-vanishing. Under certain conditions, a generalized $H^h$—birecurrent Finsler space becomes Landsberg space. Some conditions have been pointed out which reduce a generalized $H^h$—birecurrent Finsler space $F_n(n > 2)$ into Finsler space of scalar curvature.

Keywords: Finsler space; Generalized $H^h$—birecurrent Finsler space; Ricci tensor; Landsberg space; Finsler space of scalar curvature.
1. Introduction

H. S. Ruse [6] considered a three dimensional Riemannian space having the recurrent of curvature tensor and he called such space as Riemannian space of recurrent curvature. This idea was extended to n-dimensional Riemannian and non-Riemannian space by A.G. Walker [2], Y. C. Wong [18], Y.C. Wong and K. Yano [19] and others.

This idea was extended to Finsler spaces by A. Moor [3] for the first time. Due to different connections of Finsler space, the recurrent of different curvature tensor have been discussed by R.B. Misra [12], R.B. Misra and F. M. Meher [13], B.B.Sinha and S.P.Singh [4], P. N. pandey and R.B.Misra [10], R.S.D.Dubey and A.K. Srivastara [14], R.Verma [16], S. Dikshit [17], and others. S. Dikshit discussed different Finsler space with birecurrent of Cartan’s curvature tensor, birecurrent of its associate tensor and indicatrix with respect to Berwald’s and Cartan’s connections. F.Y.A.Qasem and A.A.M.Saleem [5] discussed more general Finsler space for the hv – curvature tensor satisfies the birecurrence property with respect to Berwald's coefficient Gjk and they called it UBR- Finsler space. A.A.M.Saleem [1] discussed generalized birecurrent Finsler space and U –special generalized birecurrent Finsler space. P.N.pandey, S.Saxena and A.Goswami [11] introduced a generalized H-recurrent Finsler space.

Let $F_n$ be An n-dimensional Finsler space equipped with the metric function $aF(x,y)$ satisfying the request conditions [6].

The vectors $y_i$, $y^i$ and the metric tensor $g_{ij}$ satisfies the following relations

\begin{align*}
(1.1) \quad & a) \quad y_i, y^i = F^2 \quad b) \quad g_{ij} = \delta_i y_j = \delta_j y_i \quad c) \quad y_{i|k} = 0
\end{align*}

$$
\begin{aligned}
d) \quad y^i_{|k} = 0 \quad \text{and} \quad e) \quad g_{ij|k} = 0 .
\end{aligned}
$$

Thus the unit vector $\mathbb{l}^i$ and the associate vector $l_i$ is defined by

\begin{align*}
(1.2) \quad & a) \quad \mathbb{l}^i = \frac{y^i}{F} \quad b) \quad l_i = g_{ij} \mathbb{l}^j = \delta_i F = \frac{y_i}{F} .
\end{align*}

The two processes of covariant differentiation, defined above commute with the partial

\begin{align*}
(1.3) \quad & a) \quad \delta_j \left(X^i_{|k}\right) - \left(\delta_j X^i\right)_{|k} = X^r \left(\delta_j \Gamma^i_{rk}\right) - \left(\delta_j X^i\right) P^r_{jk},
\end{align*}

\begin{align*}
& b) \quad P^r_{jk} = \left(\delta_j \Gamma^r_{kh}\right) y^h = \Gamma^{ri}_{jkh} y^h ,
\end{align*}

\begin{align*}
& c) \quad \Gamma^{si}_{jkh} y^h = G^{i}_{jkh} y^h = 0 ,
\end{align*}

\begin{align*}
& d) \quad P^{i}_{jk} y^j = 0 ,
\end{align*}

\begin{align*}
& e) \quad g^{i}_{ir} P^{i}_{kh} = P^{i}_{rkh} .
\end{align*}

The tensor $H^{i}_{jkh}$ satisfies the relation
The deviation tensor $H_k$ is positively homogeneous of degree two in $y^i$ and satisfies

(1.6) \[ H_k y^h = H_k^h, \]

(1.7) \[ H_k y^k = 0, \]

(1.8) \[ H_{jk} = H_{jki}, \]

(1.9) \[ H_k = H_{ki}, \]

and

(1.10) \[ H = \frac{1}{n-1} H^i_i. \]

where $H_{jk}$ and $H$ are called $h$-Ricci tensor [9] and curvature scalar respectively. Since contraction of the indices does not affect the homogeneity in $y^i$, hence the tensors $H_{rk}$, $H_r$ and the scalar $H$ are also homogeneous of degree zero, one and two in $y^i$ respectively. The above tensors are also connected by

(1.11) \[ H_{jk} y^j = H_k, \]

(1.12) \[ H_{jk} = \delta_j^i H_k, \]

(1.13) \[ H_k y^k = (n-1)H. \]

The tensors $H_h^i$, $H_{kh}$ and $H_{jkh}^i$ also satisfy the following:

(1.14) \[ H_{kh}^i = \delta_k^i H_h, \]

(1.15) \[ g_{ij} H_k^i = g_{ik} H_j^i. \]

The associate tensor $H_{ijkh}$ of Berwald curvature tensor $H_{jkh}^i$ is given by

(1.16) \[ H_{ijkh} = g_{jr} H_{rkh}^i. \]

The necessary and sufficient condition for a Finsler space $F_n(n > 2)$ to be a Finsler space of scalar curvature is given by

(1.17) \[ H^i_i = F^2 R(\delta_h^i - \delta^i_h) = 0. \]

A Finsler space $F_n$ is said to be Landsberg space if satisfies

(1.18) \[ y_r G_{jk}^r = -2 C_{jk|m} y^m = -2 P_{jk} = 0. \]
2. Generalized $H^h$ – Birecurrent Finsler Space

Let us consider a Finsler space $F_n$ whose Berwald curvature tensor $H^i_{jkh}$ satisfies

\[(2.1)\quad H^i_{jkh} = \lambda^i_{jkm} H^j_{kh} + \mu^i_{jkm} \left( \delta^j_k g_{jh} - \delta^j_h g_{jk} \right), \quad H^i_{jkh} \neq 0,\]

where $\lambda^i_{jkm}$ and $\mu^i_{jkm}$ are non-zero covariant vector fields and called the recurrence vector fields. Such space called a generalized $H^h$ - recurrent Finsler space.

Differentiating (2.1) covariantly with respect to $x^m$ in the sense of Cartan and using (1.1e), we get

\[(2.2)\quad H^i_{jkh;lm} = \lambda^i_{lkm} H^j_{kh} + \lambda^i_{lkm} \delta^j_k g_{jh} - \lambda^i_{lkm} \delta^j_h g_{jk}.\]

Using (2.1) in (2.2), we get

\[(2.3)\quad H^i_{jkh;lm} = (\lambda^i_{lkm} + \lambda^i_{lmk} \lambda_m^j) H^j_{kh} + \lambda^i_{lkm} \mu^j_{lmk} \delta^j_k g_{jh} - \lambda^i_{lkm} \delta^j_h g_{jk},\]

which can be written as

\[(2.4)\quad H^i_{jkh;lm} = a_{lkm} H^j_{kh} + b_{lkm} \left( \delta^j_k g_{jh} - \delta^j_h g_{jk} \right), \quad H^i_{jkh} \neq 0,\]

Where $a_{lkm} = \lambda^i_{lkm} + \lambda^i_{lmk} \lambda_m^j$ and $b_{lkm} = \lambda^i_{lkm} \mu^j_{lmk} + \mu^i_{lkm}$ are non-zero covariant tensor fields of second order and called recurrence tensor fields.

**Definition 2.1.** If Berwald curvature tensor $H^i_{jkh}$ of a Finsler space satisfying the condition (2.3), where $a_{lkm}$ and $b_{lkm}$ are non-zero covariant tensor fields of second order, the space and the tensor will be called generalized $H^h$ – birecurrent Finsler space and generalized $h$ – birecurrent tensor, respectively, we shall denote such space and tensor briefly by $GH^h - BR - F_n$ and $Gh-BR$, respectively.

However, if we start from condition (2.3), we cannot obtain the condition (2.1), we may conclude

**Theorem 2.1.** Every generalized $H^h$ – recurrent Finsler space is generalized $H^h$ – birecurrent Finsler space, but the converse need not be true.

Transvecting (2.3) by the metric tensor $g_{ir}$, using (1.1e) and (1.16), we get

\[(2.5)\quad H^i_{jkh;lm} = a_{lkm} H^j_{kh} + b_{lkm} \left( \delta^j_k g_{jh} - \delta^j_h g_{jk} \right).\]

Transvecting (2.3) by $y^j$, using (1.1d) and (1.4), we get

\[(2.6)\quad H^i_{jkh;lm} = a_{lkm} H^j_{kh} + b_{lkm} \left( \delta^j_k y_h - \delta^j_h y_k \right).\]

Further transvecting (2.5) by $y^k$, using (1.1d) and (1.6), we get

\[(2.6)\quad H^i_{jkh;lm} = a_{lkm} H^j_{kh} + b_{lkm} \left( y^j y_h - \delta^j_h F^2 \right).\]

Thus we have
Theorem 2.2. In $G^h - BR - F_n$, the associate tensor $H_{frkh}$ of Berwald curvature tensor $H_{jkh}$, the torsion tensor $H_{kh}$ and the deviation tensor $H_{h}$ are non-vanishing.

Contracting the indices $i$ and $h$ in equations (2.3), (2.5) and (2.6) and using (1.8), (1.9), (1.10) and (1.1a), we get

\begin{align}
H_{jk\ell m} &= a_{\ell m}H_{jk} + (1 - n)b_{\ell m}g_{jk} . \\
H_{k\ell m} &= a_{\ell m}H_{k} + (1 - n)b_{\ell m}y_{k} . \\
H_{i\ell m} &= a_{\ell m}H - b_{\ell m}F^2 .
\end{align}

Thus, we conclude

Theorem 2.3. In $G^h - BR - F_n$, the Ricci tensor $H_{jk}$, the curvature vector $H_{k}$ and the scalar curvature $H$ are non-vanishing.

Differentiating (2.8) partially with respect to $y^{j}$, using (1.12) and (1.1b), we get

\begin{align}
\hat{\partial}^{j}(H_{k\ell m}) &= (\hat{\partial}^{j}a_{\ell m})H_{k} + a_{\ell m}H_{jk} + (1 - n)(\hat{\partial}^{j}b_{\ell m})y_{k} \\
&\quad + (1 - n)b_{\ell m}g_{jk} .
\end{align}

Using the commutation formula exhibited by (1.3a) for $(H_{k\ell})$ and using (1.12), we get

\begin{align}
(\hat{\partial}^{j}H_{k\ell})_{im} - H_{r\ell}(\hat{\partial}^{j}\Gamma_{km}^{r}) &= -(\hat{\partial}^{j}H_{k})(\hat{\partial}^{i}_{k}\Gamma_{km}^{r}) - (\hat{\partial}^{i}_{k}H_{k})(\hat{\partial}^{j}\Gamma_{km}^{r}) \\
&\quad - (H_{k}\hat{\partial}^{j}\Gamma_{km}^{r}) - (\hat{\partial}_{r}H_{k})(\hat{\partial}^{i}_{k}\Gamma_{km}^{r}) - (\hat{\partial}^{i}_{k}H_{k})(\hat{\partial}^{j}\Gamma_{km}^{r}) \\
&\quad - (H_{k}\hat{\partial}^{j}\Gamma_{km}^{r}) = (\hat{\partial}^{j}a_{\ell m})H_{k} + a_{\ell m}H_{jk} + (1 - n)(\hat{\partial}^{j}b_{\ell m})y_{k} + (1 - n)b_{\ell m}g_{jk} .
\end{align}

Again using commutation formula exhibited by (1.3a) for $(H_{k})$ in (2.11), we get

\begin{align}
(\hat{\partial}_{i}H_{k})_{\ell} - H_{r}(\hat{\partial}_{i}\Gamma_{\ell k}^{r}) &= -(\hat{\partial}_{i}H_{k})(\hat{\partial}_{r}\Gamma_{\ell k}^{r}) - (\hat{\partial}^{i}_{r}H_{k})(\hat{\partial}_{\ell}\Gamma_{\ell k}^{r}) \\
&\quad - (H_{k}\hat{\partial}_{i}\Gamma_{\ell k}^{r}) - (\hat{\partial}_{i}H_{k})(\hat{\partial}_{r}\Gamma_{\ell k}^{r}) - (\hat{\partial}^{i}_{r}H_{k})(\hat{\partial}_{\ell}\Gamma_{\ell k}^{r}) \\
&\quad - (H_{k}\hat{\partial}_{i}\Gamma_{\ell k}^{r}) = (\hat{\partial}^{j}a_{\ell m})H_{k} + a_{\ell m}H_{jk} + (1 - n)(\hat{\partial}^{j}b_{\ell m})y_{k} + (1 - n)b_{\ell m}g_{jk} .
\end{align}

Using (1.12) and (2.7) in (2.12), we get

\begin{align}
\{-H_{r}(\hat{\partial}_{i}\Gamma_{\ell k}^{r}) - (H_{kr})(\hat{\partial}_{i}\Gamma_{\ell k}^{r})\}_{im} - H_{r\ell}(\hat{\partial}^{i}_{k}\Gamma_{km}^{r}) \\
- H_{k}(\hat{\partial}^{i}_{k}\Gamma_{km}^{r}) - (\hat{\partial}_{i}H_{k})(\hat{\partial}_{r}\Gamma_{\ell k}^{r}) - (\hat{\partial}^{i}_{r}H_{k})(\hat{\partial}_{\ell}\Gamma_{\ell k}^{r}) \\
- (H_{k}\hat{\partial}_{i}\Gamma_{\ell k}^{r}) = (\hat{\partial}^{j}a_{\ell m})H_{k} + (1 - n)(\hat{\partial}^{j}b_{\ell m})y_{k} .
\end{align}

Transvecting (2.13) by $y^{k}$, using (1.1d), (1.13), (1.3b) and (1.1a), we get
\[-2H_{r'i'j'}^r_\ell \Gamma^r_\ell m - (n - 1)H_{r'i'}(\dot{\partial}_j \Gamma^r_\ell m) = (n - 1)(\ddot{\partial}_j a_{\ell m})H - (n - 1)(\ddot{\partial}_j b_{\ell m})F^2.\]

Which can be written as

(2.14) \[
(\ddot{\partial}_j b_{\ell m}) = \frac{(\partial_j a_{\ell m})H}{p^2}. 
\]

if and only if

(2.15) \[-2H_{r'i'j'}^r_\ell \Gamma^r_\ell m - (n - 1)H_{r'i'}(\dot{\partial}_j \Gamma^r_\ell m) = 0.\]

If the tensor \(a_{\ell m}\) is independent of \(y^i\), the equation (2.14) shows that the tensor \(b_{\ell m}\) is also independent of \(y^i\).

Conversely, if the tensor \(b_{\ell m}\) is independent of \(y^i\), we get \(H\dot{\partial}_j a_{\ell m} = 0\). In view of theorem 2.3, the condition \(H\dot{\partial}_j a_{\ell m} = 0\) implies \(\dot{\partial}_j a_{\ell m} = 0\), i.e. the covariant tensor \(a_{\ell m}\) is also independent of \(y^i\). This leads to

**Theorem 2.4.** The covariant tensor \(b_{\ell m}\) is independent of the directional arguments if the covariant tensor \(a_{\ell m}\) is independent of directional arguments if and only if equation (2.15) holds.

Suppose the tensor \(a_{\ell m}\) is not independent of \(y^i\), then (2.13) and (2.14) together imply

(2.16) \[
\{-H_r(\dot{\partial}_j \Gamma^r_{\ell k}) - (H_{kr})^r_{j'} \}_m - H_{r'i'}(\partial_j \Gamma^r_{km}) \\
-H_{kr}\left(\dot{\partial}_j \Gamma^r_{\ell m} - (H_{kr})^r_{j'} \right) \} \Lambda_m - (H_{kr})^r_{j'} P^r_{j'} \Lambda_m - H_{kr} P^r_{j'} \Lambda_m \\
= (\ddot{\partial}_j a_{\ell m}) \left[H_k - \frac{(n - 1)}{p^2} H y_k \right].
\]

Transvecting (2.16) by \(y^m\) and using (1.1d), (1.3c) and (1.3d), we get

(2.17) \[
\{-H_r(\dot{\partial}_j \Gamma^r_{\ell k}) - (H_{kr})^r_{j'} \}_m y^m = (\ddot{\partial}_j a_{\ell} - a_{\ell}')(H_k - \frac{(n - 1)}{p^2} H y_k).
\]

where \(a_{\ell m} y^m = a_{\ell}\)

if

(2.18) \[-H_r(\dot{\partial}_j \Gamma^r_{\ell k}) - (H_{kr})^r_{j'} \}_m y^m = 0, \text{ equation (2.17) implies at least one of the following conditions}

(2.19) a) \(a'_{j'} = \dot{\partial}_j a_{\ell}\), b) \(H_k = \frac{(n - 1)}{p^2} H y_k\)

Thus, we have

**Theorem 2.5.** In \(GH^h - BR - F_n\) for which the covariant tensor \(a_{\ell m}\) is not independent of the directional arguments and if condition (2.18) holds, at least one of the conditions (2.19a) and (2.19b) hold.

Suppose (2.19b) holds equation (2.16) implies
\[(2.20) \quad \left\{ \frac{(n-1)}{2} H y_r \partial_j \Gamma^r_{\ell k} - H_{kr} P^r_{fj} \right\}_{im} - \left\{ \frac{(n-1)}{2} H y_r \right\}_{l\ell} \partial_j \Gamma^r_{\ell k} \]

\[- \left\{ \frac{(n-1)}{2} H y_k \right\}_{lr} \partial_j \Gamma^r_{\ell m} - H_{kr\ell} P^r_{jm} - \frac{(n-1)}{2} H y_s \left( \partial_j \Gamma^s_{\ell k} \right) P^r_{jm} \]

\[- H_{kr} P^s_{jr} P^r_{jm} = 0 . \]

Transvecting (2.20) by \( y^j \), using (1.1d), (1.3b) and (1.3d), we get

\[(2.21) \quad \left\{ \frac{(n-1)}{2} H y_r P^r_{fj} \right\}_{im} + \left\{ \frac{(n-1)}{2} H y_r \right\}_{l\ell} P^r_{jm} + \left\{ \frac{(n-1)}{2} H y_k \right\}_{lr} P^r_{jm} = 0 . \]

Thus, we have

**Theorem 2.6.** In \( GH^h - BR - F_n \), we have the identity (2.21) provided (2.19b).

Transvecting (2.21) by the metric tensor \( g_{rf} \), using (1.1e) and (1.3e), we get

\[(2.22) \quad \left\{ \frac{(n-1)}{2} H y_r P^r_{fj} \right\}_{im} + \left\{ \frac{(n-1)}{2} H y_r \right\}_{l\ell} P^r_{jm} + \left\{ \frac{(n-1)}{2} H y_k \right\}_{lr} P^r_{jm} = 0 . \]

By using (1.1c), equation (1.22) can be written as

\[
y_r \left( H P^r_{fj} \right)_{im} + y_r H_{l\ell} P^r_{jm} + y_k H_{lm} P^r_{jm} = 0 . \]

In view of theorem 2.3, we have

\[(2.23) \quad P_{fjm} = 0 . \]

if and only if

\[(2.24) \quad y_r \left( H P^r_{fj} \right)_{im} + y_r H_{l\ell} P^r_{jm} = 0 . \]

Therefore the space is Landsberg space.

Thus, we have

**Theorem 2.7.** An \( GH^h - BR - F_n \) is Landsberg space if and only if conditions (2.24) and (2.19b) hold good.

If the covariant tensor \( a_{f\ell} \neq \partial_j a_{\ell} \), in view of theorem 2.5, (2.19b) holds good. In view of this fact, we may rewrite theorem 2.7 in the following form

**Theorem 2.8.** An \( GH^h - BR - F_n \) is necessarily Landsberg space if and only if conditions (2.24) and (2.19b) hold good and provided \( a_{f\ell} \neq \partial_j a_{\ell} \).
Differentiating (2.5) partially with respect to $y_j$, using (1.5) and (1.1b), we get

\[ \partial_j \left( H_{klm}^i \right) = \left( \partial_j a_{em} \right) H_{kh}^i + a_{em} H_{jk}^i \partial_{\ell m} \left( \delta_k^i y_h - \delta_h^i y_k \right) + b_{\ell m} \left( \delta_k^i g_{jh} - \delta_h^i g_{jk} \right). \]

Using commutation formula exhibited by (1.3b) for $(H_{klm}^i)$ in (2.25), we get

\[ \partial_j \left( H_{klm}^i \right) = \partial_j a_{em} H_{kh}^i + \left( \delta_k^i b_{\ell m} \right) \left( \delta_k^i y_h - \delta_h^i y_k \right) + b_{\ell m} \left( \delta_k^i g_{jh} - \delta_h^i g_{jk} \right). \]

Again applying the commutation formula exhibited by (1.3a) for $(H_{klm}^i)$ in (2.26) and using (1.5), we get

\[ \partial_j \left( H_{klm}^i \right) = \partial_j a_{em} H_{kh}^i + \partial_j b_{\ell m} \left( \delta_k^i y_h - \delta_h^i y_k \right) + b_{\ell m} \left( \delta_k^i g_{jh} - \delta_h^i g_{jk} \right). \]

Using (2.3) in (2.27), we get

\[ \partial_j \left( H_{klm}^i \right) = \partial_j a_{em} H_{kh}^i + \partial_j b_{\ell m} \left( \delta_k^i y_h - \delta_h^i y_k \right) + b_{\ell m} \left( \delta_k^i g_{jh} - \delta_h^i g_{jk} \right). \]

Transvecting (2.28) by $y_k^i$, using (1.1d), (1.1a), (1.3b), (1.4) and (1.6), we get

\[ \partial_k \left( H_{klm}^i \right) = \partial_k a_{em} H_{kh}^i + \partial_k b_{\ell m} \left( \delta_k^i y_h - \delta_h^i y_k \right) + b_{\ell m} \left( \delta_k^i g_{jh} - \delta_h^i g_{jk} \right). \]
\[-H^i_{r(t)}(\partial_j\Gamma^r_{tm}) - H^i_{h(t)}(\partial_j\Gamma^r_{nt}) - (H^i_{rh(t)} + H^i_k(\partial_j\Gamma^r_{st})) - H^i_j(\partial_i\Gamma^r_{ht})\]

\[-2H^i_{sh}P^r_{jt}P^r_{jm} = (\partial_ja_{\ell m})H^i_{h(t)} + (\partial_jb_{\ell m})(y^i-y^j - \delta^i_jF^2)\].

Substituting the value of \(\partial_jb_{\ell m}\) from (2.14), in (2.29), we get

\[(2.30)\quad \{H^i_{h(t)}(\partial_j\Gamma^r_{nt}) - H^i_{h(t)}(\partial_j\Gamma^r_{nt}) - 2H^i_{rh}P^r_{jt} + H^i_{h(t)}(\partial_j\Gamma^r_{st}) - H^i_j(\partial_i\Gamma^r_{ht})\]

\[-H^i_{r(t)}(\partial_j\Gamma^r_{tm}) - H^i_{h(t)}(\partial_j\Gamma^r_{nt}) - (H^i_{rh(t)} + H^i_k(\partial_j\Gamma^r_{st})) - H^i_j(\partial_i\Gamma^r_{ht})\]

\[-2H^i_{sh}P^r_{jt}P^r_{jm} = (\partial_ja_{\ell m})(H^i_{h(t)} - H(\delta^i_j - \|\nabla\rangle)).\]

if

\[(2.31)\quad \{H^i_{h(t)}(\partial_j\Gamma^r_{nt}) - H^i_{h(t)}(\partial_j\Gamma^r_{nt}) - 2H^i_{rh}P^r_{jt} + H^i_{h(t)}(\partial_j\Gamma^r_{st}) - H^i_j(\partial_i\Gamma^r_{ht})\]

\[-H^i_{r(t)}(\partial_j\Gamma^r_{tm}) - H^i_{h(t)}(\partial_j\Gamma^r_{nt}) - (H^i_{rh(t)} + H^i_k(\partial_j\Gamma^r_{st})) - H^i_j(\partial_i\Gamma^r_{ht})\]

\[-2H^i_{sh}P^r_{jt}P^r_{jm} = 0 .\]

We have at least one of the following conditions :

\[(2.32)\quad a) \quad (\partial_ja_{\ell m}) = 0 , \quad b) \quad H^i_{h(t)} = H(\delta^i_j - \|\nabla\rangle).\]

Putting \(H = F^2R\), the equation (2.32b) may be written as

\[(2.33) \quad H^i_{h(t)} = F^2R(\delta^i_j - \|\nabla\rangle),\]

where \(R \neq 0\). Therefore the space is a Finsler space of scalar curvature .

Thus , we have

**Theorem 2.9.** An \(GH^h - BR - F_n\) for \(n > 2\) admitting equation (2.31) holds is a Finsler space of scalar curvature provided \(R \neq 0\), the covariant tensor \(a_{\ell m}\) is not independent of directional arguments .

**3. Conclusions**

(3.1) The space whose defined by condition (2.3) is called generalized \(H^h -\) birecurrent Finsler space.

(3.2) Every generalized \(H^h -\) recurrent Finsler space is generalized \(H^h -\) birecurrent Finsler space, but the converse need not be true.
(3.3) In generalized $H^b$ – birecurrent Finsler space the Berwald curvature tensor $H_{jkh}$ and the associate tensor $H_{jrhk}$ satisfies the generalized birecurrence property.

(3.4) The torsion tensor $H_{kh}^i$, the deviation tensor $H_{hi}^i$, the Ricci tensor $H_{jk}$, the vector $H_k$ and the scalar curvature tensor $H$ are all non-vanishing in our space.

(3.5) An generalized $H^b$ – birecurrent Finsler space is necessarily Landsberg space if and only if conditions (2.24), (2.19b) and $a_\ell \neq \partial_\ell a_\ell$ hold.

(3.6) An generalized $H^b$ – birecurrent Finsler space for $n > 2$ is a Finsler space of scalar curvature provided $R \neq 0$, the covariant tensor $a_\ell m$ is not independent of directional arguments and condition (2.31) holds.

4. Recommendations

The authors recommend that the research should be continued in the Finsler spaces because it has many applications in in relativity physics and other fields.

References

283-288.


