A Generalization of Notion Group as Dynamical Groups

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Abstract

In this paper the concept of groups will be extended by a dynamical system to the dynamical groups and we will investigate some results about them. Also notions coset and quotient dynamical group are introduced.

Keywords: Dynamical groups; group; quotient dynamical group.

1. Introduction

Some generalizations of notion group are presented so far. For example generalized group is introduced by Molaie [2] and is studied in [1-6]. We assume the reader is familiar with the definition of dynamical system [7]. In this paper we introduce a new generalization of group by notion dynamical system that we call dynamical group. The paper is organized as follows. In Section 2 the notion of dynamical groups is introduced, also we define dynamical subgroups and give some properties and examples about dynamical groups. In Section 3 we define and study notions coset and quotient dynamical groups.

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2. Dynamical groups

**Definition 2.1.** Let $G$ be a non-empty set and $\circ$ be an associative operation on $G$. We say $(G, f, g, h, \circ)$ is a dynamical group if the functions $f: G \to G$, $g: G \to G$ and $h: G \to G$ be injective and satisfies the following conditions:

i) There exists $e \in G$ such that for every $x \in G$,

$$f(x) o g(e) = f(e) o g(x) = h(x), \quad (1)$$

Which $e$ is called the identity element of $(G, f, g, h, \circ)$

ii) For every $x \in G$ there exists $x' \in G$ such that

$$f(x) o g(x') = f(x') o g(x) = h(e), \quad (2)$$

Which $x'$ is called the inversion element of $x$ in $(G, f, g, h, \circ)$. Note that every function of the functions $f, g, h$ is considered as a dynamical system.

**Remark 2.2.** The notion of dynamical groups is a generalization of groups. In fact if $f = g = h$ and the functions are injective and surjective then $(G, \circ)$ is a group because if $x \in G$ and $e = f(e)$ then there exists $\bar{x} \in G$ such that $x = f(\bar{x})$ and we can write

$$\bar{e} o x = f(e) o f(\bar{x})$$

$$= f(e) o g(\bar{x})$$

$$= h(\bar{x}) = f(\bar{x}) = x, \quad (3)$$

Also

$$x o \bar{e} = f(\bar{x}) o f(e)$$

$$= f(\bar{x}) o g(e)$$

$$= h(\bar{x}) = f(\bar{x}) = x. \quad (4)$$

Now let $y \in G$. Since $f$ is surjective there exists $x \in G$ such that $y = f(x)$. By the property (2), there exists $x' \in G$ such that: (let $y' = f(x')$)

$$y o y' = f(x) o f(x') = f(x) o g(x')$$
\[ h(e) = f(e) = e \quad (5) \]

\[ y'oy = f(x')of(x) = f(x')og(x) = h(e) = f(e) = e. \quad (6) \]

So \((G, o)\) is a group.

**Example 2.3.** Let \(G\) be an abelian group and \(a, b \in G\). If \(f(x) = ax, g(x) = b \cdot x\) and \(h(x) = a \cdot b \cdot x\). Then we have

\[ f(x) \cdot g(e) = a \cdot x \cdot (b \cdot e) = a \cdot x \cdot b = a \cdot b \cdot x = h(x), \quad (7) \]

\[ f(e) \cdot g(x) = a(e) \cdot (b \cdot x) = a \cdot b \cdot x = h(e). \quad (8) \]

where \(e\) is the identity element of \(G\).

On the other hand let \(x \in G\) and \(x'\) be the Inversion element of \(x\) in \(G\), then we can write

\[ f(x) \cdot g(x') = a \cdot x \cdot b \cdot x' = a \cdot b \cdot x' = a \cdot b \cdot e = h(e), \quad (9) \]

\[ f(x') \cdot g(x) = a \cdot x' \cdot b \cdot x = a \cdot b \cdot x' \cdot x = a \cdot b \cdot e = h(e). \quad (10) \]

Hence \((G, f, g, h)\) is a dynamical group.

**Example 2.4.** Let \(G = \{a, b, c\}\). We define an operation on \(G\) by

\[ aa = a, \ ab = b, \ ac = a, \]

\[ ba = a, \ bb = c, \ bc = b, \]

\[ ca = c, \ cb = a, \ cc = c. \quad (11) \]

Also let the functions \(f, g, h\) are defined by

\[ f(a) = b, \ f(b) = a, \ f(c) = c, \]

\[ g(a) = b, \ g(b) = c, \ g(c) = a, \]

\[ h(a) = b, \ h(b) = a, \ h(c) = c. \quad (12) \]
Then \( (G, f, g, h) \) is a dynamical group because

i) \( b \) is the identity element because

\[
\begin{align*}
  f(a)g(b) &= bc = ab = f(b)g(a) = b = h(a), \\
  f(b)g(b) &= ac = f(h(b))g(b) = a = h(b), \\
  f(c)g(b) &= cc = ca = f(b)g(c) = c = h(c). \\
\end{align*}
\] (13)

ii) Inverse element: we can write

\[
\begin{align*}
  a' &= c, & b' &= b, & c' &= a
\end{align*}
\]

because

\[
\begin{align*}
  f(a)g(a') &= ba = cb = f(a')g(a) = a = h(b), \\
  f(b)g(b') &= ac = f(b')g(b) = a = h(b), \\
  f(c)g(c') &= cb = ba = f(c')g(c) = a = h(b). \\
\end{align*}
\] (14)

**Proposition 2.5.** Let \( (G, f, g, h) \) be a dynamical group, then \( (x')' = x \).

**Proof.** It can be deduced from property 2.2.

**Definition 2.6.** Let \( (G, f, g, h) \) be a dynamical group. A non-empty subset \( H \) of \( G \) is called a dynamical subgroup of \( G \) if \( (H, f_H, g_H, h_H, o) \) be a dynamical group.

Since \( f, g, h \) are injective, \( f_H, g_H, h_H \) are injective.

**Definition 2.7.** We say two dynamical groups \( (G, f, g, h, o), (\bar{G}, \bar{f}, \bar{g}, \bar{h}, \bar{o}) \) are isomorphic if there exists a bijective function \( \phi: G \rightarrow \bar{G} \) such that

i) \( \phi(xo'y) = \phi(x)\phi(y) \),

ii) \( \phi of = \bar{f}o\phi, \phi og = \bar{g}o\phi, \phi oh = \bar{h}o\phi \).

Then \( \phi \) is called an isomorphism.

**Proposition 2.8.** Let \( (G, f, g, h, o), (\bar{G}, \bar{f}, \bar{g}, \bar{h}, \bar{o}) \) be two dynamical groups and \( \phi: G \rightarrow \bar{G} \) be an isomorphism. Then
i) If \( e \) is the identity element of \( (G, f, g, h, o) \), then \( \phi(e) \) is the identity element of \( (\bar{G}, \bar{f}, \bar{g}, \bar{h}, \bar{o}) \).

ii) If \( x' \) is the inversion element of \( x \) in \( (G, f, g, h, o) \), then \( \phi(x') \) is the inversion element of \( \phi(x) \) in \( (\bar{G}, \bar{f}, \bar{g}, \bar{h}, \bar{o}) \).

**Proof.**

i) By definition 2.1 it is sufficient to show that

\[
\bar{f}(x)\bar{g}(\phi(e)) = \bar{f}(\phi(e))\bar{g}(x) = h(x). \tag{15}
\]

From definition 2.7 we have

\[
\bar{f}(x)\bar{g}(\phi(e)) = \bar{f}(x)\phi(g(e)) = \phi(f(\phi^{-1}x))\phi(g(e)) = \phi(h(\phi^{-1}x)) = (\phi o h o \phi^{-1})(x) = h(x). \tag{16}
\]

Also

\[
\bar{f}(\phi(e))\bar{g}(x) = \phi(f(e))\phi(g(\phi^{-1}x)) = \phi(f(e))\phi(g(\phi^{-1}x)) = \phi(h(\phi^{-1}x)) = (\phi o h o \phi^{-1})(x) = h(x). \tag{17}
\]

ii) We show that

\[
\bar{f}(\phi(x))\bar{g}(\phi(x')) = \bar{f}(\phi(x'))\bar{g}(\phi(x)) = h(\phi(e)). \tag{18}
\]

We can write

\[
\bar{f}(\phi(x))\bar{g}(\phi(x')) = \phi(f(x))\phi(g(x')) = \phi(f(x)g(x')) = \phi(h(e)) = (\phi o h o \phi^{-1})(\phi(e))
\]
\[
= \tilde{h}(\varphi(e)). \quad (19)
\]

Also
\[
\tilde{f}(\varphi(x'))\tilde{g}(\varphi(x)) = \varphi(f(x'))\varphi(g(x))
\]
\[
= \varphi(f(x'g(x)) = \varphi(h(e))
\]
\[
= (\varphi\circ h\circ \varphi^{-1})(\varphi(e)) = \tilde{h}(\varphi(e)). \quad (20)
\]

3. Coset and quotient dynamical group

Definition 3.1. Let \((G, f, g, h, o)\) be a dynamical group. Also let \(H\) be a dynamical subgroup of \(G\) and \(c \in G\). We define left and right cosets of \(H\) in \(G\) as
\[
cH = \{f(c)g(h)|h \in H\}, \quad (21)
\]
\[
Hc = \{f(h)g(c)|h \in H\}. \quad (22)
\]

Remark 3.2. Let \(H\) be a dynamical subgroup of dynamical group \((G, f, g, h, o)\) and \(e\) be the identity element of \(G\). Then we have
\[
eH = He = h(H). \quad (23)
\]

Because
\[
eH = \{f(e)g(x)|x \in H\} = \{f(x)g(e)|x \in H\}
\]
\[
= He = \{h(x)|x \in H\} = h(H). \quad (24)
\]

Lemma 3.3. Let \(H\) be a dynamical subgroup of dynamical group \((G, f, g, h)\). Also let for each \(x, y \in G\)
\[
f(x)g(y) = g(y)f(x),
\]
\[
f(xy) = f(x)f(y),
\]
\[
g(xy) = g(x)g(y). \quad (25)
\]
Then for every \( c, b \in G \)

\[(cH)(bH) = (cb)H. \quad (26)\]

**Proof.**

\[
(cH)(bH) = \{f(c)g(h)|h \in H\}\{f(b)g(h')|h' \in H\}
\]

\[
= \{f(c)g(h)f(b)g(h')|h, h' \in H\}
\]

\[
= \{f(c)f(b)g(h)g(h')|h, h' \in H\}
\]

\[
= \{f(cb)g(hh')|h, h' \in H\}
\]

\[
= (cb)H. \quad (27)
\]

**Proposition 3.4.** Let \( H \) be a dynamical subgroup of dynamical group \((G, f, g, h)\), such that Also let for each \( c \in G \), \( cH = Hc \). Also let for every \( x, y \in G \),

\[
f(x)g(y) = g(y)f(x)
\]

\[
f(xy) = f(x)f(y)
\]

\[
g(xy) = g(x)g(y). \quad (28)
\]

Then the family of left (right) cosets of \( H \) in \((G, f, g, h)\) which is shown by \((G/H, f', g', h')\) is a dynamical group called the quotient dynamical group of \( G \) by \( H \) such that \( f', g', h' \) are defined by

\[
f'(cH) = f(c)H,
\]

\[
g'(cH) = g(c)H,
\]

\[
h'(cH) = h(c)H. \quad (29)
\]

**Proof.**

It is necessary to check the axioms of dynamical group on \((G/H, f', g', h')\). \( f', g', h' \) are injective functions. If \( e \) is the identity element in \((G, f, g, h)\), then \( eH \) is the identity element in \((G/H, f', g', h')\) because

\[
f'(cH)g'(eH) = f(c)Hg(e)H
\]
\[ f(c)g(e)H = h(c)H = h'(cH), \quad (30) \]

\[ f'(eH)g'(cH) = f(e)Hg(c)H = f(e)g(c)H = h(c)H = h'(cH). \quad (31) \]

Now let \( c' \) be the inverse of \( c \) in \((G, f, g, h)\). We show that \( c'H \) is the inverse of \( cH \) in \((G/H, f', g', h')\).

\[ f'(cH)g'(c'H) = f(c)Hg(c'H) = f(c)g(c'H) = h(e)H = h'(eH), \quad (32) \]

\[ f'(c'H)g'(cH) = f(c'H)g(c)H = f(c')g(c)H = h(e)H = h'(eH). \quad (33) \]

So the axioms of dynamical group are satisfied in \((G/H, f', g', h')\).

**Example 3.5.** Let \((G, f, g, h)\) be the dynamical group of example 2.3 with \( aa = a \) and \( bb = b \). Also let \( H \) be a dynamical subgroup of dynamical group \( G \) such that for each \( c \in G \), \( cH = Hc \). Then \((G/H, f', g', h')\) is a quotient dynamical group because by proposition 3.4 we can write

\[ f(x)g(y) = (ax)(by) = (ay)(bx) = g(y)f(x), \quad (34) \]

\[ f(xy) = axy = aaxy = (ax)(ay) = f(x)f(y), \quad (35) \]

\[ g(xy) = bxy = bbxy = (bx)(by) = g(x)g(y). \quad (36) \]
4. Conclusion

We introduced the notion dynamical groups as a generalization of concept group. Also we defined notions coset and quotient dynamical group and we presented some propositions and examples about them.

References


